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CAYLEY GRAPHS HAVING NICE ENUMERATIONS

BY

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ABSTRACT

Let W be the Cayley graph of an infinite finitely generated group and M be a finite cover of W. It is proved in the paper that $Th(M)$ is finitely axiomatizable over W if W has a nice enumeration (in the sense of G. Ahlbrandt and M. Ziegler). A finitely generated free abelian group provides such an example. It is shown that in the non-abelian case the corresponding examples are rather rare. In particular, in the soluble case they must be virtually abelian. We discuss the finite model property for finite covers of Cayley graphs of virtually abelian groups and the existence of nice enumerations for strongly minimal structures in general.

0. Introduction

Let W be a transitive structure of bounded valency (for example, a coloured locally finite graph), Γ be its infinite component and $G = Aut(\Gamma)$. Let $\pi: \Gamma_1 \to \Gamma$ be a finite cover of Γ (see [7] or definitions below), where Γ_1 is a transitive structure of bounded valency. Then π induces a surjective homomorphism $\rho: G_1 = Aut(\Gamma_1) \to G$. The main questions that we consider are as follows. *When is the finite cover* Γ_1 *finitely axiomatizable over the base* Γ *? When does* Γ_1 *have the finite model property* (for any sentence ϕ holding in Γ_1 there is a finite structure M such that $M \models \phi$) ?

Questions of this kind frequently arise in model theory. Several very deep results were obtained (mainly by Ahlbrandt, Ziegler and Hrushovski) for many ω-categorical structures (see [2], [14] and [4]). The present paper is an attempt to extend these results to the non-ω-categorical case.

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In some sense finite covers of structures of bounded valency have a group-theoretic origin. Any homomorphism of finitely generated groups ρ: $G_1 \to G$ with finite kernel can be considered as a finite cover of the Cayley graph of G. Any typical question arising in this situation (splitting, '$\mathfrak{R}$-by-finite = finite-by-$\mathfrak{R}$?') naturally extends to a question on finite covers. For example, for finitely presented G and G_1, the possibility of lifting residual finiteness from G to G_1 is equivalent to the statement that the finite model property for the Cayley graph of G implies the finite model property for the Cayley graph of G_1 (see Section 4).

A well-known theorem of P. Hall states that a finite-by-nilpotent group is nilpotent-by-finite (virtually nilpotent) [9]. In particular, finitely generated finite-by-nilpotent groups are residually finite (by another theorem of Hall finitely generated virtually abelian-by-nilpotent groups are residually finite [10]). Does a finite cover of the Cayley graph of a nilpotent group have the finite model property ? The confirmation can be considered as a version of Hall's theorem for infinitely generated groups. We shall prove in Section 4 that a finite cover with finite kernel of the Cayley graph of a finitely generated nilpotent group has the finite model property. Moreover, if the group is virtually $\cong \mathbf{Z}^2$, then the assumption of kernel finiteness can be removed. Notice that in the latter case the automorphism group of the cover is not necessarily finitely generated.

Throughout the paper the case of Cayley graphs of virtually abelian groups is in the centre of our study. The reason comes from the fact that they (and their finite covers) are the only known non-ω-categorical structures with a *nice enumeration*. That is the key notion of the paper.

Nice enumerations were introduced by G. Ahlbrandt and M. Ziegler in [2] and were based on the approach of G. Higman from [12]. In Section 2 we prove that if π: $M \to W$ is a finite cover of bounded valency and M has a nice enumeration, then there exists a sentence Φ such that $Th(M)$ is axiomatized by $\{\Phi\} \cup Th(W)$. In Section 4 we apply this theorem to finite covers of Cayley graphs of abelian groups. We show that if a cover admits a good notion of *envelope* (as in [5]), then it has a covering expansion with finite kernel.

The existence of envelopes is a strong version of the finite model property. In Section 4 we give some examples which show that an envelope is not easy to construct (even in the case of finite covers with finite kernels). We also show that a nice enumerable structure may have a finite cover with infinite kernel and without proper covering expansions. This can be considered as a strong version of *non-splitting*; the situation cannot be reduced to the cases of *superlinked* covers and *free* covers as frequently happens for ω-categorical structures [7].

In Section 3 structures of bounded valency without nice enumerations are obtained. This shows that there are cases where the most natural (and still unique) approach to relative finite axiomatisability does not work. We give the dividing line in the case of Cayley graphs of virtually solvable groups: only Cayley graphs of virtually abelian groups have nice enumerations. The proof involves some non-trivial algebraic material due J. H. Wolf [28].

It is worth noting that nice enumerations have become a notion interesting independently from relative finite axiomatizability. D. Evans was the first who considered them in full generality (and without the assumption of ω-categoricity). In [3] he together with A. Camina gives a useful connection with permutation modules. Applications to chain conditions in covers and to 'Ziegler's Finiteness Conjecture' (even in the non-ω-categorical case) can be found in [6] and [7].

It is shown in [18] that any disintegrated strongly minimal structure of a finite language can be considered as a structure of bounded valency. Thus our results work for disintegrated strongly minimal structures in general. In Section 5 we study what happens if the strongly minimal structure is not disintegrated. We prove that a locally modular strongly minimal group having a nice enumeration is ω-categorical. Applying some valuation theory we prove a similar statement in the case of affine spaces over division rings. We also discuss the non-locally modular case.

In some sense the results of the paper are negative. It turns out that the only known non-ω-categorical, strongly minimal structures with a nice enumeration are finite covers of Cayley graphs of virtually abelian groups. The results of Sections 3 and 5 show that it is very difficult to obtain any new example; in general, finite covers of strongly minimal structures should be studied by methods not applying nice enumerations. On the other hand, already in the case of Cayley graphs of virtually abelian (nilpotent) groups problems of quasi (relative) finite axiomatizability look very deep and involve a lot of algebraic material. The results of Section 4 suggest that some questions usually arising in the ω-categorical case (splitting and finiteness of cohomology groups) become natural here.

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1. Structures of bounded valency and their finite covers

Let W be a transitive ω-saturated strongly minimal structure. Let $\pi\colon M \to W$ be a surjection such that the equivalence relation E_π induced by π is definable in M. Assume that the set of all relations 0-definable in W coincides with the set of all relations on M/E_π which are 0-definable in M (here W is identified with M/E_π). In this case we say that M is a **finite cover** of W if $Aut(M)$ induces $Aut(W)$ on W and all *fibres* $M(w) = \pi^{-1}(w)$ are finite. Then it is clear that the E_π-classes are of the same size and the structure M is almost strongly minimal. The group of automorphisms fixing W pointwise, $Aut(M/W)$, is called the **kernel of the cover** $Ker(\pi)$. Since the E_π-classes are finite, the kernel is profinite. If the kernel is finite we say that the cover is **superlinked**. The cover **splits** if there is a closed subgroup in $Aut(M)$ covering $Aut(W)$ and having trivial intersection with the kernel. We now give examples which are central in the paper.

Example: Let G be an infinite group generated by $\{g_1, \ldots, g_n, \ldots\}$. We consider the Cayley graph of G as a binary structure $\Gamma_G = (G, R_1, \ldots, R_n, \ldots)$ where $(a,b) \in R_i \leftrightarrow ag_i = b$. Then $Aut(\Gamma_G) = G$ under the left action. Let D be a finite set and $\Gamma = \Gamma_G \times D$. We consider Γ as a two-sorted structure in a language consisting of the relations of Γ_G, the projection $\pi\colon \Gamma \to \Gamma_G$ and finitely many binary relations defining a (coloured) locally finite graph. We require that any automorphism of Γ_G extends to an automorphism of Γ. The disjoint union of countably many copies of this structure forms an ω-saturated almost strongly minimal model of $Th(\Gamma, \Gamma_G)$ which is a finite cover of the strongly minimal saturated structure W consisting of countably many copies of Γ_G.

It is worth noting that the relations R_i can be considered as unary functions and then it is easy to see that $Th(\Gamma_G)$ has quantifier elimination. The latter implies that if $Th(\Gamma_G)$ admits a finite language then G is finitely generated.

On the other hand, notice that any homomorphism of finitely generated groups $\rho\colon G_1 \to G$ with finite kernel creates such a cover $\Gamma_{G_1} \to \Gamma_G$ in a finite language and with finite kernel. We may claim that finite covers of finitely generated groups (with a finite language) form a natural family of group extensions located between the properties *finite-by-(finitely generated)* and *profinite-by-(finitely generated)*. Then the following questions look central.

Let $\pi\colon \Gamma \to \Gamma_G$ be a finite cover as above. *Is there an expansion of the structure Γ which is a cover Γ_G with a finite kernel (in other words, is the profinite-by-(finitely generated) case reducible to the finite-by-(finitely generated case)) ? When does the group extension $Aut(\Gamma) \to G$ split ?* We shall show in Section 4 that the solutions to these questions depend on the situation.

1.1. Structures of bounded valency. The main results of the paper are formulated for structures of bounded valency. The definition is as follows. Let N be an infinite structure of a finite relational language L. We say that N is of **bounded valency** if there is a natural number k such that for any $r \in L$ and $a \in N$ the size of the set $\{\bar{b} \subset N: a \in \bar{b}, \bar{b} \models r(\bar{x})\}$ is not greater than k.

For $a, b \in N$ we can define the **distance** $d(a, b)$ to be the least number n such that there are tuples $\bar{c}_1, \ldots, \bar{c}_n$ realizing basic relations of L such that $a \in \bar{c}_1, b \in \bar{c}_n$ and for every $i < n$ the set $\bar{c}_i \cap \bar{c}_{i+1}$ is not empty. A **component** of N is a maximal subset A such that the distance between any two elements of A is defined. For $C_1, C_2 \subset N$ we define $d(C_1, C_2) = min(d(c_1, c_2): c_1 \in C_1, c_2 \in C_2)$.

For $\bar{a} = (a_1, \ldots, a_n) \subset N$ the t-**ball** $B(t, \bar{a})$ is the set of all $b \in N$ such that $d(\bar{a}, b) \leq t$. All t-balls are finite and described as L-structures by so-called **spherical formulas:** $\phi(\bar{x})$ is **spherical** with respect to t if it describes the isomorphism type of the t-ball of $\bar{x}$ as an L-structure with constants corresponding to $\bar{x}$. The following proposition has been proved in [11, 21].

Proposition 1.1: (1) *Any formula is equivalent in $Th(N)$ to a Boolean combination of spherical formulas.*

(2) *Given a finite language L and natural numbers s and n there are natural numbers $J_1(s, n)$ and $J_2(s, n)$ such that if L-structures N and N' have valency not greater than s and for any isomorphism type τ of $J_1(s, n)$-balls of points they satisfy the same sentences of the form "there is a set of the size i of $J_1(s, n)$-balls of the type τ that are distant from each other by at least $J_2(s, n)$", $i \leq n + 1$, then the structures N and N' satisfy the same sentences containing not greater than n quantifiers.*

Note that if N is strongly minimal and non-ω-categorical then all non-algebraic components in N are infinite, transitive and pairwise isomorphic. Since a totally categorical theory is not finitely axiomatisable we have the following corollary.

Corollary 1.2: *Let N be ω-saturated, strongly minimal and finitely axiomatizable. Then there is a natural number t such that the theory of a non-algebraic component in N is axiomatized by the sentence describing the t-ball of a point.*

1.2. Finite covers of structures of bounded valency. If $M \to W$ is a finite cover of a structure of bounded valency, then the connected components of W induce a partition of M. In the proposition below we describe when M is of bounded valency and this partition consists of the connected components of M.

PROPOSITION 1.3: *Let $\pi: M \to W$ be a finite cover of a strongly minimal transitive structure W. Assume that W and M are structures of finite languages.*

(1) *If M is a structure of bounded valency then W admits a finite language of bounded valency.*

(2) *If W is a structure of bounded valency then there is a set L of relations of bounded valency 0-definable in M which induce the relations of W and have the property that the type (in M) of any tuple of elements of the same L-component in M is determined by formulas from L.*

(3) *Let W be of bounded valency and L be as in (2). If M is ω-saturated and any two M-automorphisms of any two L-components of M have a common extension in $Aut(M)$ then M is a structure of some finite language $L_0 \subset L$.*

Proof: (1) Let M be a structure of bounded valency. W.l.o.g. we may assume that M is ω-saturated. We consider M in the language of all spherical formulas $\phi(\bar{x})$ such that if $\phi(\bar{x})$ describes the t-ball of $\bar{x}$ then it asserts that all elements of $\bar{x}$ are in the t-ball of one of them. Now to every relation R_i of the language of M we associate the relation $\pi(R_i)$. The latter is 0-definable in W and has bounded valency. Since W is ω-saturated, it suffices to show that any finite map in W preserving all formulas in the language of the relations $\pi(R_i)$ is elementary in W.

If $\bar{a}, \bar{b} \subset W$ satisfy the same formulas of the form $\pi(R_i)$ then by the definition of formulas R_i, for any t, the t-balls of $M(\bar{a})$ and $M(\bar{b})$ ($M(\bar{a})$ denotes $\pi^{-1}(\bar{a})$) are isomorphic under a map taking $\bar{a}$ onto $\bar{b}$. By Proposition 1.1 this means that some enumerations of $M(\bar{a})$ and $M(\bar{b})$ agreeing with the tuples $\bar{a}$ and $\bar{b}$ have the same types in M. Since M is a cover of W, the tuples $\bar{a}$ and $\bar{b}$ have the same type in W.

(2) Let W be a structure of bounded valency and consider W in the language of all spherical formulas defined as in (1) (for all $t \in \omega$). Then to every relation R_i of bounded valency definable in W we associate a finite sequence $r_{i,j}$, $1 \leq j \leq l_i$, of all possible types of $Th(M)$ over $\emptyset$ such that if $\bar{b} \models r_{i,j}$ then there exists $\bar{a} \models R_i$ such that $\bar{b} \subset M(\bar{a})$. Since M is a finite cover and W is transitive, any 1-type of M is determined by a formula. Since any R_i is of bounded valency the same is true for all $r_{i,j}$, $1 \leq j \leq l_i$. So any $r_{i,j}$ is determined by a formula. It is easily seen that the corresponding formulas form a required language L.

(3) Let $\bar{a}$ and $\bar{b}$ from M have the same length and the corresponding subtuples satisfy the same formulas of the form $r_{i,j}$. By Proposition 1.1 the tuples $\pi(\bar{a})$ and $\pi(\bar{b})$ have the same type in W. Since M is a cover of W there are enumerations of $M(\pi(\bar{a}))$ and $M(\pi(\bar{b}))$ which have the same types in M. Now using (2) and

the assumptions of (3) we get that $\bar{a}$ and $\bar{b}$ have the same type in M. Since M is ω-saturated and has finite language we can find a language L_0 as in the statement of (3). ∎

It follows from Proposition 1.3(1) and a remark in the beginning of Section 1 that if a finite cover of the Cayley graph of a group G admits a finite language of bounded valency then G is finitely generated.

From now on we consider covers $\pi: M \to W$, where M, W are defined as in Proposition 1.3(3). The reason is that we are interested in the situation when M is of bounded valency. This is the case of the following cover. Let Γ be an infinite component of an ω-saturated structure W of bounded valency and $\pi_\Gamma: \hat{\Gamma} \to \Gamma$ be a finite cover of Γ where $\hat{\Gamma}$ also is a structure of bounded valency. If M is an ω-saturated structure of the theory $Th(\hat{\Gamma})$ then π_Γ induces a cover $\pi: M \to W$ satisfying the assumptions of Proposition 1.3(3).

1.3. Construction. We now give a construction producing a cover for any structure of bounded valency. This provides natural examples of covers. Moreover, the construction will be applied later in the paper.

Let W be a transitive structure of bounded valency in a finite language L and t be a natural number. We assume that any component of W is infinite. Let $w \in W$ and $\hat{w}$ be an enumeration of $B(t,w)$, where w is the first. Let W^p be the set of all tuples realising $p = tp(\hat{w}/\emptyset)$. Define $\pi: \hat{w} \to w$. For any relation R of W we define $\hat{R}$ on W^p to be $\pi^{-1}(R)$. We also include in the language the equivalence relation E_π on W^p induced by π and the relation $I(\hat{w}, \hat{w}_1, \ldots, \hat{w}_k)$ asserting that $\hat{w} = (w, w_1, \ldots, w_k)$. Then W^p becomes a structure of bounded valency. The map π induces a finite cover of W. It is easily seen that the kernel of π is trivial.

Lemma 1.4: (1) *If W is axiomatizable by the type of its t-balls then W^p is axiomatizable by the type of its 1-balls.*

(2) *The structure W^p is finitely axiomatizable if and only if W is finitely axiomatizable.*

Proof: (1) Let M have the same 1-balls as W^p. If $S_{\hat{v}}$ is the set of all M-elements I-adjacent to $\hat{v}$, then the relations $\hat{R}$ define a structure on $S_{\hat{v}}/E_\pi$ isomorphic to the structure in W^p defined in the same way. The latter contains a t-ball of an element of W. This shows that $Th(M/E_\pi) = Th(W)$. A sequence $\hat{v}, \hat{v}_1, \ldots, \hat{v}_k \in S_{\hat{v}}$ realising I in M defines an enumeration of the t-ball of the element $\hat{v}/E_\pi$ in M/E_π. Since M and W^p have the same 1-balls, the relation I is recovered by this enumeration and the R-structure of M/E_π. Thus $Th(M) = Th(W^p)$.

(2) Let s be a natural number such that if the s-balls of points in a structure W' are isomorphic to $B(s,w), w \in W$; then $Th(W') = Th(W)$. It is easy to see that if a structure M has the same s-balls as W^p then $Th(M/E_\pi) = Th(W)$. As above this guarantees that $Th(M) = Th(W^p)$.

The necessity follows from the construction of W^p in W. ∎

We now assume that the stabilizer $Aut(W/w)$ is finite. Then there is a number t such that any isomorphism from $B(t,w)$ onto itself has a unique extension in $Aut(W/w)$. The action of $Aut(W)$ on the corresponding structure W^p has trivial point-stabilizer.

Let $l := \mid \pi^{-1}(w)\mid$. Notice that if a structure W' has the same $(2t+1)$-balls as W, then there is a natural structure (denoted by $(W')^p$) defined on $W' \times \{1,\ldots,l\}$ of the language of W^p with the same 1-balls as W^p. Indeed, let π on $(W')^p$ be the projection onto W' and E_π be the corresponding equivalence relation. For a relation R of W' define $\hat{R} := \pi^{-1}(R)$. For any $w' \in W'$ we fix a 1-1 correspondence between all p-enumerations of $B(t,w')$ and $\pi^{-1}(w')$. Then we define the relation I as in the case of W^p. Taking an isomorphism from $B(2t+1,w')$ onto $B(2t+1,w)$ one can see that the relation I on the set of all tuples meeting $\pi^{-1}(w')$ is a copy of the corresponding relation on W^p. Similar arguments prove the following lemma.

LEMMA 1.5: *Let W, t and p be as above. If a structure W' has the same $(2tl+1)$-balls as W, then the structures W^p and $(W')^p$ have the same l-balls. If a structure M has the same l-balls as W^p, then M/E_π and W have the same l-balls.*

2. Nice enumerations and finite axiomatizability over the base

2.1. NICE ENUMERATIONS. The following notion was introduced in [2] (in a slightly different form) as a technical tool for finite covers of strictly minimal ω-categorical structures.

> An enumeration $\{w_0, w_1, \ldots\}$ of a permutation structure (G,W) ($G = Aut(W)$) is an **AZ-enumeration** if the set $\Sigma = \{(w,S) : (w,S)$ is a G-conjugate of some pair $(w_n, \{w_0,\ldots,w_{n-1}\})\}$ (the set of *nice pairs*) satisfies the following conditions:
>
> (1) any $P \subset \Sigma$ contains a finite set P_0 such that for any $(w,S) \in P$ there are $(w',S') \in P_0$ and $\alpha \in G$ such that $\alpha(S') \subseteq S$ and $\alpha(w') = w$;

(2) for all k there is k' such that between any pair of sets $T \subset S$, where S is *nice* (i.e., (w, S) is a nice pair for some w) and $|T| < k$, we can find a nice set S' with $|S'| < k'$.

We say that an enumeration of W is **nice** if it satisfies the first condition of the definition of AZ-enumerations. It is clear that this condition is equivalent to the condition that any $P \subset \Sigma$ does not have an infinite anti-chain with respect to the quasiorder defined by:

$$(w', S') \leq^* (w, S) \leftrightarrow (\exists \alpha \in G)(\alpha(w') = w \wedge \alpha(S') \subseteq S).$$

The definition implies that a structure realizing infinitely many 1-types (for example, having infinite $acl(\emptyset)$) does not have a nice enumeration. It is worth noting that a finite cover $C \to W$ of a structure having a nice enumeration, has a nice enumeration too (see Section 7.2.2 of [7]).

It is easily seen that the existence of an AZ-enumeration implies ω-categoricity of W. However, it is not hard to find a non-ω-categorical structure having a nice enumeration (see Section 3). On the other hand, in order to apply nice enumerations to finite covers of non-ω-categorical structures (using the method of [2]) we want nice enumerations to satisfy some variants of (2). In the case of structures of bounded valency it can be done as follows.

We say that an enumeration of (G, W) is a **regular** enumeration if for all k there is k' such that between any pair of sets $S' \subset S$, where (w, S) is a nice pair and S' is a subset of the k-ball of w, we can find a nice set T of size $< k'$ such that (w, T) is a nice pair.

Fortunately nice enumerations are regular.

PROPOSITION 2.1: *Let W be a transitive connected structure of bounded valency. Then any nice enumeration of W is regular.*

Proof: Let $w_0, w_1, \ldots$ be a nice enumeration. Suppose that the enumeration is not regular. Then there exists a natural number k and an infinite sequence of triples $(A_i \cup \{v_i\}, S_i, T_i)$ with $|T_0| < |T_1| < \cdots$, such that (v_i, S_i) is a nice pair, $A_i \subseteq B(k, v_i) \cap S_i$ and T_i is a smallest subset of S_i such that $A_i \subseteq T_i$ and (v_i, T_i) is a nice pair.

By boundedness of valency we may assume that all A_i are of the same size and can be enumerated as $\bar{a}_i$ such that the tuples $v_i\bar{a}_i$ have the same type. Since the sequence (v_i, T_i) consists of nice pairs of a nice enumeration, one can find a subsequence increasing with respect to the quasi-order $\leq^*$. We may now assume that the sequence (v_i, T_i) is linearly ordered by $\leq^*$.

Let l be the maximal distance from v_i to an element of A_i. By boundedness of valency we may assume that for any $i < j$, if α embeds T_i into T_j and takes v_i to v_j, then α maps $T_i \cap B(l, v_i)$ onto $T_j \cap B(l, v_j)$. Then any $\bar{c} \in T_j$ of the type $tp(\bar{a}_j/v_j)$ has a preimage under α.

Let $n \in \omega$. Suppose that for any $i < j \leq n$ any α as above does not take $\bar{a}_i$ to $\bar{a}_j$. Then T_{n-1} contains a tuple $\bar{c}$ (a preimage of $\bar{a}_n$) such that for any $i < n-1$ any α realising the embedding $(v_i, T_i) \leq^* (v_{n-1}, T_{n-1})$ does not take $\bar{a}_i$ to $\bar{c}$. Taking an appropriate embedding of T_{n-2} into T_{n-1} we find $\bar{d}_1$ and $\bar{d}_2 \in T_{n-2}$ (the preimages of $\bar{c}$ and $\bar{a}_{n-1}$) such that for any $i < n-2$ any α realising the embedding $(v_i, T_i) \leq^* (v_{n-2}, T_{n-2})$ does not take $\bar{a}_i$ to $\bar{d}_1$ or $\bar{d}_2$.

This procedure creates n pairwise distinct tuples in $T_0 \cap B(l, v_0)$. By boundedness of valency we now see that there are $i < j$ and an embedding $\alpha : (v_i, T_i) \to (v_j, T_j)$ which takes $\bar{a}_i$ to $\bar{a}_j$. Since $|\alpha(T_i)| < |T_j|$, we obtain a contradiction with the definition of T_j. ∎

2.2. Finite axiomatizabity over the base. We now describe the situation of the main result of this section. Let W be a non-ω-categorical strongly minimal transitive structure of bounded valency and let Γ be its infinite component. Let $G = Aut(\Gamma)$ and E be a G-invariant finite equivalence relation on Γ (not necessarily definable) and G_0 be the subgroup of G consisting of all automorphisms fixing the E-classes. Let Γ_0 be an E-class. Assume that G_0 is transitive on Γ_0 and there are 0-definable in W relations $\vec{R}'$ of bounded valency such that Γ_0 is a connected component under these relations and G_0 is the automorphism group of the corresponding structure on Γ_0. In this case we say that Γ is a **finite extension of** $(\Gamma_0, \vec{R}')$. Note that G_0 is normal in G and its index is finite. The main reason for this definition is the following example.

Example: Let G be an infinite group generated by $\{g_1, \ldots, g_n\}$. Consider the Cayley graph $\Gamma_G = (G, R_1, \ldots, R_n)$ where $G = Aut(\Gamma_G)$ under the left action. Let $G_0 < G$ be a normal subgroup of finite index generated by $\{g'_1, \ldots, g'_m\}$ and $\Gamma_0 = \Gamma_{G_0}$ be the corresponding Cayley graph under $R'_1, \ldots, R'_m$ which are defined by $(a, b) \in R'_i \leftrightarrow ag'_i = b$. Then Γ_G is a finite extension of Γ_0 and the corresponding equivalence relation E consists of cosets of G_0 in G.

Theorem 2.2: *Let W be a non-ω-categorical, countable, saturated, transitive, strongly minimal structure of bounded valency. Let π: $M \to W$ be a finite cover of W and E_π be the corresponding equivalence relation (0-definable in M).*

Assume that M is a structure of bounded valency under the language induced by W as in Proposition 1.3(3). Let an infinite component Γ of W be a finite extension of a structure Γ_0 and let Γ_0 have a nice enumeration. Then there is a sentence $\phi \in Th(M)$ such that $Th(M)$ is axiomatizable by $Th(W) \cup \{\phi\}$.

Proof: We apply the strategy of the corresponding theorem from [2]. Let $G = Aut(\Gamma)$ and $G_0 = Aut(\Gamma_0)$. Let E be the equivalence relation determined by G and G_0. Fix a nice enumeration $<$ of Γ_0 and the corresponding set Σ of nice pairs. Enumerate all E-classes of W in the same way up to appropriate isomorphisms. In the following lemma we consider elements of Γ_0 with respect to the structure M.

LEMMA 2.3: *There is a finite number λ such that for all $(w, S) \in \Sigma$ there is a subset $S' \subset S$ satisfying $|S'| < \lambda$, $(w, S') \in \Sigma$ and for $\bar{a} = \pi^{-1}(w)$*

(i) $tp(\bar{a}/\pi^{-1}(S') \cup \{w\})$ implies $tp(\bar{a}/\pi^{-1}(S) \cup \{w\})$,

(ii) $tp(w/\pi^{-1}(S'))$ implies $tp(w/\pi^{-1}(S))$, and $tp(w/S')$ implies $tp(w/S)$.

Proof: Notice that $w \in acl(S)$ (by boundedness of valency). By the definition of a nice enumeration there are finitely many $(w_i, S_i) \in \Sigma$ such that any nice pair contains some (w_i, S_i) under an appropriate automorphism (as in the definition of $\leq^*$). This shows that for all nice pairs (w, S) the distance $d(w, S)$ is bounded by a fixed number and by boundedness of valency the number of conjugates of w over S is also bounded. Now the statement becomes Lemma 4.3 from [2]. The proof in [2] uses only the definition of a nice enumeration and works in our case. ∎

Since the relations of Γ and Γ_0 are of bounded valency, there exists a number q such that the q-ball of any element of Γ contains an element from Γ_0. Also note that for any $a, b \in \Gamma$ we have $B(q, ab) = B(q, a) \cup B(q, b)$. This implies that for any $(w, S) \in \Sigma$ the number $|B(q, wS) \setminus B(q, S)|$ is bounded by $|B(q, w)|$. The following lemma is a slight modification of Lemma 2.3.

LEMMA 2.4: *The number λ from Lemma 2.3 can be chosen so that for all $(w, S) \in \Sigma$ there is a subset $S' \subset S$ satisfying $|S'| < \lambda$, $(w, S') \in \Sigma$, $B(q, wS) \setminus B(q, S) = B(q, wS') \setminus B(q, S')$ and for $\bar{w} = B(q, wS) \setminus B(q, S)$ and $\bar{b} = \pi^{-1}(\bar{w})$*

(i) $tp(\bar{b}/\pi^{-1}(B(q, S')) \cup \bar{w})$ implies $tp(\bar{b}/\pi^{-1}(B(q, S)) \cup \bar{w})$,

(ii) $tp(\bar{w}/\pi^{-1}(B(q, S')))$ implies $tp(\bar{w}/\pi^{-1}(B(q, S)))$, and $tp(\bar{w}/B(q, S'))$ implies $tp(\bar{w}/B(q, S))$.

We now define a new language for M (denoted by L^*). For all $(w, S) \in \Sigma$ such that $|S| < \lambda$ we introduce new relational symbols R_p corresponding to all types p such that $M \models p(\bar{a}_1, \ldots, \bar{a}_k)$ implies that there are $w_1, \ldots, w_l \in W$ such that $(w_l, \{w_1, \ldots, w_{l-1}\})$ is a conjugate of (w, S) and for some enumeration $w_1, \ldots, w_k$ of $B(q, \{w_1, \ldots, w_l\})$ we have $\bigwedge_i \bar{a}_i = \pi^{-1}(w_i)$. Then we interpret R_p by p. Notice that R_p is definable in M and the number of such types p is finite. Indeed, $tp(S \cup \{w\})$ is isolated, because it is of rank 1 and W is transitive. Then the type p must be of rank 1 and by the same reason is isolated. It is clear that R_p is of bounded valency. Now the finiteness of the set of types p as above is obvious.

We also assume that L^* contains π and the language of W. Let M^* be the structure M with respect to L^*.

Let M_1 and M_2 be two L^*-structures both having W as the range of π. A **$*$-map** σ between M_1 and M_2 is a partial isomorphism defined on a set of the form $\pi^{-1}(B(q,S)) \cup B(q,S)$ such that σ maps $B(q,S)$ elementarily with respect to W and S is conjugate with some $S' \subset \Gamma_0$ with $(w, S') \in \Sigma$ for some w.

LEMMA 2.5: *Any $*$-map between M^* and M^* is M-elementary.*

Proof: We proceed by induction on the size of S. For S with $|S| \leq \lambda$ we have the lemma by the definition of L^*. The rest can be done by an application of Lemma 2.4 and the argument of Case 2 of the proof of Lemma 4.4 from [2]. ∎

We can now show that M *and* M^* *are interdefinable.* Indeed, M^* is definable in M by the definition of M^*. To prove the converse take an automorphism α of M^*. Let $\hat{\Gamma} = \pi^{-1}(\Gamma)$. Then as α is elementary on W we have that α maps $\hat{\Gamma}$ onto some component of M. Since Γ is the union of an increasing chain of sets $B(q,S)$ with $(w,S) \in \Sigma$, the component $\hat{\Gamma}$ is the union of an increasing chain of the form $\pi^{-1}(B(q,S)) \cup B(q,S)$. So, α on $\hat{\Gamma}$ is an increasing union of $*$-maps. By Lemma 2.5, α is elementary on $\hat{\Gamma}$. This implies that α is elementary on every component of M. Thus α is an automorphism of M. Since M is saturated we have that M and M^* are interdefinable.

We now state the axioms of M^*. First note that by boundedness of valency there exists a number k_0 such that if

(a) (w, S') is a nice pair from Γ_0 with $|S'| < \lambda$,

(b) S'' is a G-conjugate of a nice set consisting of $\leq \lambda$ elements,

(c) $A \subset B(2q, S'')$ and $w \in A$,

then $A \cup S'$ *is contained in the* k_0*-ball of* w.

Choose μ as k' in the definition of a regular enumeration for $k = k_0$ and let μ' be the maximal size of the q-balls in Γ of the corresponding μ-element subsets

$Tw \subset \Gamma_0$ as in that definition.

We say that a formula of M^* is **$*$-quantifier free** if it is a boolean combination of quantifier free formulas and formulas describing the types of $B(q,S) \subset \Gamma$ such that $(w,S) \in \Sigma$. Now extend the axioms of W and π by the following ones:

(0) The axiom that asserts that a structure is of the valency of M^* and describes the type of any $\pi(R_p)$ in the language of W (by an apppropriate isolating formula).

(1) All sentences of the form

$$\forall \bar{x}_1,\ldots,\bar{x}_{\mu'},w_1,\ldots,w_{\mu'}\phi(\bar{x}_1,\ldots,\bar{x}_{\mu'},w_1,\ldots,w_{\mu'})$$

which are true in M^*, where ϕ is $*$-quantifier-free.

(2) All sentences of the form

$$\forall \bar{x}_1,\ldots,\bar{x}_l,w_1,\ldots,w_l\exists \bar{y}_1,\ldots,\bar{y}_{l'},w'_1,\ldots,w'_{l'}\phi(\bar{x}_1,\ldots,\bar{x}_l,\bar{y}_1,\ldots,\bar{y}_{l'},\bar{w},\bar{w}')$$

which are true in M^*, where ϕ is $*$-quantifier-free,

$$l \leq max(|\ B(q,S)|\ : (w',S) \in \Sigma, |\ S|\ < \lambda) \quad \text{and} \quad l' \leq |\ B(q,w)|\ , \quad w \in \Gamma_0.$$

It is clear that the number of axioms in (0), (1), (2) is finite. It remains to check that M^* is axiomatizable by them modulo $Th(W)$.

Let M_1^* be a model of these axioms such that W is the base of M_1^*. We want to show that M^* and M_1^* are isomorphic. It suffices to show that their components of the form $\pi^{-1}(\Gamma)$ are isomorphic. Let $\hat{\Gamma}$ and $\hat{\Gamma}_1$ be the corresponding components in M^* and M_1^* respectively. We shall use the back-and-forth argument. (Note that by boundedness of valency forth suffices in our case.) Take the first $w_0 \in \Gamma_0$ under the enumeration. The axioms of (2) guarantee the existence of $w'_0 \in \Gamma$ such that $(\pi^{-1}(B(q,w_0)),B(q,w_0))$ and $(\pi^{-1}(B(q,w'_0)),B(q,w'_0))$ (viewed as appropriate tuples) realise the same $*$-quantifier-free formulas in M^* and M_1^* respectively. Note that by transitivity and (1), we can take $w'_0 = w_0$. The rest of our argument is contained in the following lemma.

LEMMA 2.6: *Let $*$-map σ between $\hat{\Gamma}$ and $\hat{\Gamma}_1$ be defined on $\pi^{-1}(B(q,S)) \cup B(q,S)$ for some $(w,S) \in \Sigma$. Then σ can be extended to a $*$-map defined on $\pi^{-1}(B(q,Sw)) \cup B(q,Sw)$.*

Proof: Choose $S' \subset S$ as in Lemma 2.4. Let $S' = \{w_1,\ldots,w_k\}$, $B(q,S') = \{w_1,\ldots,w_l\}$, $\bar{a}_i = \pi^{-1}(w_i)$ and $B(q,S'w) \setminus B(q,S') = \{w'_1,\ldots,w'_{l'}\}, \bar{b}_i = \pi^{-1}(w'_i)$.

CLAIM 1: *The restriction of σ on $B(q,S') \cup \pi^{-1}(B(q,S'))$ has a prolongation τ to $B(q,S'w) \cup \pi^{-1}(B(q,S'w))$.*

Indeed, let ϕ_1 describe the $*$-qf type of $\bar{a}_1,\dots,\bar{a}_l,w_1,\dots,w_l$ and ϕ_2 describe the $*$-qf type of

$$\bar{a}_1,\dots,\bar{a}_l,\bar{b}_1,\dots,\bar{b}_{l'},w_1,\dots,w_l,w'_1,\dots,w'_{l'}.$$

It is easily seen that

$$\forall \bar{x}_1,\dots,\bar{x}_l,v_1,\dots,v_l \exists \bar{y}_1,\dots,\bar{y}_{l'},v'_1,\dots,v'_{l'}(\phi_1 \to \phi_2)$$

is an axiom from (2) (apply Lemma 2.5). This together with

$$M_1^* \models \phi_1(\sigma(\bar{a}_1),\dots,\sigma(\bar{a}_l),\sigma(w_1),\dots,\sigma(w_l))$$

gives the claim.

CLAIM 2: *$\sigma \cup \tau$ is a $*$-map defined on $B(q,Sw) \cup \pi^{-1}(B(q,Sw))$.*

By Lemma 2.4 the type $tp(\bar{w}'/B(q,S'))$ implies $tp(\bar{w}'/B(q,S))$. This provides that $\sigma \cup \tau$ is elementary on $B(q,Sw)$. It remains to show that $\sigma \cup \tau$ is compatible with the predicates R_p. By the definition of R_p it suffices to show R_p-compatibility for any restriction of $\sigma \cup \tau$ on $\pi^{-1}(B(q,S''))$ where $B(q,S'') \subset B(q,Sw)$ and S'' is of cardinality not greater than λ and is a G-conjugate of some initial segment of the nice enumeration.

Since $B(q,Sw) = \bigcup\{B(q,x) : x \in S \cup \{w\}\}$, we find $A \subset (S \cup \{w\}) \cap B(2q,S'')$ such that $B(q,S'') \subseteq B(q,A)$. If $w \notin A$ then $B(q,S'') \subset B(q,S)$ and is in the domain of σ. Therefore we assume that $w \in A$. Then the set $S' \cup A$ is in the k_0-ball of w (see the definition of k_0). By the choice of μ there is $T \subseteq S$ such that $S' \cup A \subseteq T \cup \{w\}$ and (w,T) is a nice pair of size $\leq \mu$. Now it suffices to show that the restriction of $\sigma \cup \tau$ on $\pi^{-1}(B(q,Tw)) \cup B(q,Tw)$ preserves all $*$-quantifier-free formulas.

Let ϕ_2 be as in the previous claim. Let ϕ_3 describe the $*$-qf type of $\pi^{-1}(B(q,Tw)) \cup B(q,Tw)$ and ϕ_4 be the corresponding formula describing the $*$-qf type of $\pi^{-1}(B(q,T)) \cup B(q,T)$. Note that if a subset $\pi^{-1}(B(q,T'w')) \cup B(q,T'w')$ realises (as an appropriate tuple) $\phi_2 \wedge \phi_4$ in M^* then the corresponding map

$$\rho\colon \pi^{-1}(B(q,Tw)) \cup B(q,Tw) \to \pi^{-1}(B(q,T'w')) \cup B(q,T'w'))$$

is elementary. Indeed, if ρ_1 is its restriction on $\pi^{-1}(B(q,S'w) \cup B(q,S'w)$ and ρ_2 is its restriction on $\pi^{-1}(B(q,T)) \cup B(q,T))$ then both ρ_1 and ρ_2 are elementary by Lemma 2.5. Now by the choice of S' we see that ρ is elementary. This implies that ϕ_3 is realised by $\pi^{-1}(B(q,T'w')) \cup B(q,T'w')$. As a result we have that $\phi_2 \wedge \phi_4 \to \phi_3$ is an axiom from (1). Since this sentence holds in M_1^* and τ and σ preserve ϕ_2 and ϕ_4 respectively, the map $\sigma \cup \tau$ on $\pi^{-1}(B(q,Tw)) \cup B(q,Tw)$ preserves all $*$-qf formulas. ∎

To finish the proof of the theorem note that if M_0^* is a model of the above axioms such that $\pi(M_0^*)$ is countably saturated as a $Th(W)$-model, then $\pi(M_0^*)$ is isomorphic to W. By Lemma 2.6 the components of M_0^* are isomorphic to the components of M^*. Thus M^* and M_0^* are isomorphic. ∎

3. Nice Cayley enumerations

In this section we discuss the existence of a nice enumeration of a Cayley graph. The case of virtually abelian groups is the starting point of our analysis. Some extra evidence why this case is so important is provided by the observation below.

This concerns the ascending chain condition in the permutation modules associated with permutation groups. The following notions can be found in [3].

Let (G,Ω) be a permutation group. Let $F\Omega$ be the left F-vector space with basis Ω and FG be the F-algebra (the group algebra over F) obtained by extending the multiplication on G to the whole of FG. Then $F\Omega$ becomes a left FG-module by extending the action $(ag)w = a(gw)$ (where $ag \in FG$ and $w \in \Omega$) to the whole of FG.

PROPOSITION 3.1: *Let $G = \langle g_1,\dots,g_n\rangle$ be a finitely generated group and F be a field such that the group ring FG does not have the ascending chain condition for left ideals. Then the corresponding Cayley graph $(\Gamma_G, R_1,\dots,R_n)$ (defined on G by $(w,w') \in R_i \leftrightarrow wg_i = w'$) does not have a nice enumeration. In particular, if G contains an infinite chain of subgroups: $H_0 < H_1 < \cdots$ then Γ_G does not have a nice enumeration.*

Proof: The set $F\Gamma_G$ is naturally identified with FG. Then each left ideal of FG becomes a FG-submodule of $F\Gamma_G$. Thus $F\Gamma_G$ does not have the ascending chain condition for submodules. On the other hand, Theorem 2.4 from [3] states that the existence of a nice enumeration implies the ascending chain condition for submodules.

If G contains a chain as above and F is a field then for each $i \in \omega$ define

$$V_i = \{\Sigma_{j\in\delta} a_j w_j \in FG : (\forall k \in \delta)(\Sigma\{a_j\colon w_j H_i = w_k H_i\} = 0)\}.$$

Clearly any V_i is a left ideal of FG. It is easily seen that $V_i \subset V_{i+1}$ and $V_i \neq V_{i+1}$. This implies that the ascending chain condition for left ideals in FG does not hold. ∎

An obvious corollary of this proposition is that any Gayley graph of a free group does not have a nice enumeration. Another corollary is that if the Cayley graph of a solvable group $G = \langle g_1, \ldots, g_n \rangle$ has a nice enumeration then G is virtually polycyclic [8]: there is a polycyclic normal subgroup of finite index. At this point it is worth noting the following statement (which seems to be folklore).

Lemma 3.2: *Let a finitely generated group H be a normal subgroup of finite index of a finitely generated group G. Then Γ_G has a nice enumeration if and only if Γ_H has such an enumeration.*

Proof: It is easily seen that any enumeration of Γ_H induced by a nice enumeration of Γ_G is nice (here we only use the assumption that H is a subgroup of G). To see the contrary direction choose representatives of the cosets G/H: $g_0, \ldots, g_k$. Now order G in the following way: $hg_i < h'g_j$ $(h, h' \in H)$ if and only if h is less than h' under the nice enumeration of H or $h = h'$ and $i < j$. The rest is by straightforward aguments. ∎

3.1. Nice enumerations and virtually abelian groups. The discussion above shows that the Cayley graphs of virtually abelian finitely generated groups form a natural class where the existence of a nice enumeration should be verified (they form the simplest class of virtually polycyclic groups). The following statement describes the case of a cyclic group.

Proposition 3.3: *For any finite sequence of integers, $z_1, \ldots, z_n \in \mathbf{Z}$, generating the group $\mathbf{Z}$, the Cayley graph of $\langle z_1, \ldots, z_n \rangle$ has a nice enumeration. The graph $\Gamma_{\mathbf{Z}}$ with respect to a free generator ($z = 1$) has an enumeration which is not nice.*

Proof: First, we define a nice enumeration of $\mathbf{Z} = \langle 1 \rangle$ as a function from ω onto $\mathbf{Z}$. For $n \in \omega$ let $\nu_{\mathbf{Z}}(n)$ be $(n-1)/2 + 1$ if n is odd and $-n/2$ if n is even. It is easily verified that $\nu_{\mathbf{Z}}$ is a nice enumeration of the Cayley graph determined by the generator 1 of $\mathbf{Z}$.

Consider the general case. We may assume that $0 < z_n < \cdots < z_1$ (other cases can be treated similarly). To each $z \in \mathbf{Z}$ we associate a sequence of integers $\bar{m}_z = (m_1, \ldots, m_n)$ such that $z = \Sigma m_i z_i$ and $\Sigma | m_i |$ (denoted by $\Sigma | \bar{m}_z |$ below) is minimal. We also require that for any $(k_1, \ldots, k_n)$ such that $z = \Sigma k_i z_i$ and

$\Sigma|m_i| = \Sigma|k_i|$ the tuple $(m_1, \ldots, m_n)$ is not greater than $(k_1, \ldots, k_n)$ in the lexicographical ordering induced by $\nu_{\mathbf{Z}}$.

Note that there is a number l' such that any $z \in \mathbf{Z}$ can be decomposed as $\Sigma k_i z_i$, where $|k_i| < l'$ for $i > 1$. This provides a number l $(\geq l')$ such that for any $z \in \mathbf{Z}$ and $i > 1$, $|m_i| < l$, where $\bar{m}_z = (m_1, \ldots, m_n)$.

Now define an enumeration of $\mathbf{Z}$ as follows. Put $z < u$ if $\Sigma|\bar{m}_z| \leq \Sigma|\bar{m}_u|$ and if these sums are equal then the tuple $\bar{m}_z$ is less than $\bar{m}_u$ in the lexicographical ordering induced by $\nu_{\mathbf{Z}}$.

To show that this enumeration is nice note that for any infinite $A \subseteq \mathbf{Z}$ there is an infinite $C \subseteq A$ such that for any $z, u \in C$ if $\bar{m}_z = (m_1, \ldots, m_n)$ and $\bar{m}_u = (k_1, \ldots, k_n)$ then $m_i = k_i$ for $i \geq 2$ and m_1 and k_1 are of the same sign (apply Ramsey's theorem and the definition of the number l). Then in the case $|k_1| < |m_1|$ the element $(m_1 - k_1)z_1$ takes the initial segment determined by u into the initial segment determined by z.

To define an enumeration which is not nice, take natural numbers $a_n, n \geq 1$, defined by $a_n = \Sigma_{k=1}^{n} k$. We can now define an enumeration μ as follows. On the one hand, we require that μ is a bijection from any $\{0, \ldots, 2a_n\}$ onto $\{-a_n, \ldots, a_n\}$. On the other hand, we put $\mu(x) < \mu(y)$ for $2a_n < y < x \leq 2a_{n+1}$. It is easily seen that the elements $a_n \in \mathbf{Z}, n \in \omega$, define an antichain of initial segments under the enumeration μ. ∎

Remark 3.4: Notice that the nice enumeration constructed in the proof satisfies the following property: for any k and z with a sufficiently large $m_1 \in \bar{m}_z$, the intersection of $B(k, z)$ and the initial segment determined by z is contained in the $(m_1 - k_1)z_1$-conjugate of the initial segment determined by u, where $\bar{m}_u = (k_1, \ldots, k_n)$, $|k| \cdot (l \cdot n) < |k_1|$, m_1 and k_1 are of the same sign and $m_i = k_i$ for $i \geq 2$. Then the number k' from the definition of a regular enumeration can be chosen as the size of the longest initial segment determined by an element u with $|k_1| = |k| \cdot (l \cdot n) + 1$.

The proposition shows that the Cayley graph of a group can have both a nice enumeration and an enumeration which is not nice. This suggests that we must restrict ourselves by only 'natural' enumerations of Cayley graphs.

Definition: We say that an enumeration $\{w_0, w_1, \ldots\}$ of a structure (G, W) of bounded valency is **natural** if for all $i < j$ we have $d(w_0, w_i) \leq d(w_0, w_j)$.

It is clear that the nice enumeration defined in Proposition 3.3 is natural.

THEOREM 3.5: *Let A be a free abelian group freely generated by $(a_1, \ldots, a_n) \in A^n$. Then there exists a natural nice enumeration of the Cayley graph of $(A, a_1, \ldots, a_n)$.*

Proof: We will use the enumeration $\nu_{\mathbf{Z}}$ defined in Lemma 3.3. Now define an enumeration ν_A of A as follows. Put

$$\nu_A^{-1}(a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}) < \nu_A^{-1}(a_1^{l_1} \cdot \ldots \cdot a_n^{l_n})$$

if $\Sigma\{|\, k_i| \;: 1 \leq i \leq n\} \leq \Sigma\{|\, l_i| \;: 1 \leq i \leq n\}$ and if these sums are equal then the tuple $(k_1, \ldots, k_n)$ is less than $(l_1, \ldots, l_n)$ in the lexicographical order induced by $\nu_{\mathbf{Z}}$. The obtained enumeration is a natural enumeration. To see that this enumeration is nice note that by Ramsey's theorem any infinite set $I \subseteq A$ of words of the form $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$ contains an infinite subset J such that for any two words $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$ and $a_1^{l_1} \cdot \ldots \cdot a_n^{l_n}$ from J the following conditions hold:

$$(1) \bigwedge_{1 \leq i \leq n} ((0 < l_i \leftrightarrow 0 < k_i) \wedge (0 = l_i \leftrightarrow 0 = k_i)),$$

$$(2) \bigwedge_{1 \leq i,j \leq n} (|\, k_i| \; < |\, k_j| \; \leftrightarrow |\, l_i| \; < |\, l_j|\,).$$

Then we may also assume that J satisfies the following condition.

> (3) Consider the indexes $1 \leq t \leq n$ with the property that J contains infinitely many words $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$ with pairwise distinct k_t. Then for any pair of such indexes i and j and the words of J as above the following condition holds: $(|\, k_i| \; < |\, l_i| \; \leftrightarrow |\, k_j| \; < |\, l_j|\,))$.

Now easy computations show that for any two words $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$ and $a_1^{l_1} \cdot \ldots \cdot a_n^{l_n}$ satisfying conditions (1), (2) and $(\forall 1 \leq i \leq j \leq n)(|\, k_i| \; \leq |\, l_i| \; \leftrightarrow |\, k_j| \; \leq |\, l_j|\,))$, the nice pair determined by $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$ is embeddable (with respect to the quasi-order $\leq^*$ on Σ introduced in Section 2) into the nice pair determined by $a_1^{l_1} \cdot \ldots \cdot a_n^{l_n}$. The corresponding automorphism must be $a_1^{l_1 - k_1} \cdot \ldots \cdot a_n^{l_n - k_n}$. Of course, this guarantees that the enumeration is nice. ∎

Remark 3.6: As we noticed in Section 2 the theorem implies that the enumeration ν_A is regular. We now estimate the number k' from the definition of a regular enumeration in the case of ν_A.

Let $k \in \omega$. Let J_k be the set of all words which occur in the nice enumeration not later than the words of the form $a_1^{k \cdot m_1} \cdot \ldots \cdot a_n^{k \cdot m_n}$, where $-n^2 \leq m_i \leq n^2$ for

$i \le n$. We claim that the number k' can be chosen to be the size of the longest initial segment determined by an element from J_k.

Notice that any nice pair $(a_1^{l_1} \cdot \ldots \cdot a_n^{l_n}, S)$ contains (with respect to the quasi-order $\le^*$) a conjugate of a nice pair $(a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}, S')$ with $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n} \in J_k$. To prove this it suffices to notice that required $k_i, i \le n$, can be chosen to be a tuple satisfying $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n} \in J_k$, $| k_i| \le | l_i|$ for $1 \le i \le n$ and conditions (1), (2) from the proof of the theorem. Also note that we can take k_i such that $k_i = l_i$ for $| l_i| \le k$, and $| k_i| - | k_j| = | l_i| - | l_j|$ if $| | l_i| - | l_j| | \le k$, and $| | k_i| - | k_j| | > k$ otherwise.

If $a_1^{s_1} \cdot \ldots \cdot a_n^{s_n} \in S$ (where S is as above) is in the k-ball of $a_1^{l_1} \cdot \ldots \cdot a_n^{l_n}$ then $\Sigma\{|l_i - s_i| : 1 \le i \le n\} \le k$. Thus under the above choice of k_i we have $\Sigma\{| k_i - (k_i - l_i + s_i)| : 1 \le i \le n\} \le k$ and the element $\Pi a_i^{k_i - l_i + s_i}$ is in the k-ball of $a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}$. It remains to prove that this element is in S'.

First note that if $k < | l_i|$ then $k < | k_i|$ and the inequalities of the previous paragraph imply that $| l_i - s_i| \le k$. In particular, l_i (and k_i) is positive if and only if s_i is positive. On the other hand, if $| l_i| \le k$ then $l_i = k_i$ by the choice of k_i.

CASE 1: $\Sigma| s_i| < \Sigma| l_i|$. Then $\Sigma| k_i| - \Sigma| k_i - l_i + s_i| = \Sigma(| k_i| - | k_i - l_i + s_i|)$. We must show that the latter sum is positive. As we noted above, for $| l_i| \le k$ we have $| k_i| - | k_i - l_i + s_i| = | l_i| - | s_i|$. If $k < | l_i| \ge | s_i|$ then

$$| k_i| - | k_i - l_i + s_i| = | k_i| - | k_i| + | l_i - s_i| = | l_i| - | s_i|.$$

If $k < | l_i| < | s_i|$ then

$$| k_i| - | k_i - l_i + s_i| = | k_i| - | k_i + s_i| + | l_i| = | l_i| - | s_i|.$$

As a result we have

$$\Sigma(| k_i| - | k_i - l_i + s_i|) = \Sigma(| l_i| - | s_i|) > 0.$$

CASE 2: $\Sigma| s_i| = \Sigma| l_i|$. Fix the first i_0 such that $\nu_{\mathbf{Z}}^{-1}(s_{i_0}) < \nu_{\mathbf{Z}}^{-1}(l_{i_0})$. By the definition of ν_A we have $s_i = l_i$ and $k_i = k_i - l_i + s_i$ for $i < i_0$. If $| l_{i_0}| \le k$ then $k_{i_0} = l_{i_0}$ and $\nu_{\mathbf{Z}}^{-1}(k_{i_0} - l_{i_0} + s_{i_0}) < \nu_{\mathbf{Z}}^{-1}(k_{i_0})$. Thus $\Pi a_i^{k_i - l_i + s_i} \in S'$.

If $k < | l_{i_0}|$ then by the definition of $\nu_{\mathbf{Z}}$ we have $| s_{i_0}| < | l_{i_0}|$. Since $| l_{i_0} - s_{i_0}| \le k$ and $k < | k_{i_0}|$, we have that $k_{i_0} - l_{i_0} + s_{i_0}$ and k_{i_0} are of the same sign, and the absolute value of the former element is less than the absolute value of the latter. This guarantees that $\nu_{\mathbf{Z}}^{-1}(k_{i_0} - l_{i_0} + s_{i_0}) < \nu_{\mathbf{Z}}^{-1}(k_{i_0})$ and $\Pi a_i^{k_i - l_i + s_i} \in S'$.

Theorem 3.5 provides the following application of the results of the previous section.

COROLLARY 3.7: *Any finite cover of the Cayley graph of a finitely generated virtually abelian group is finitely axiomatizable over the base.*

From Theorem 3.5 and Lemma 3.2 we also have the following corollary.

COROLLARY 3.8: *The Cayley graph of a virtually abelian finitely generated group has a nice enumeration.*

3.2. NICE CAYLEY ENUMERATIONS. In this section we discuss how strong the condition of nice enumerability is in the case of Cayley graphs of finitely generated groups. In general we have the following conjecture (its form is motivated by the fact that enumerations produced in Lemma 3.2 are not always natural).

CONJECTURE: *Let G be a finitely generated group. If for any tuple $\bar{g} \in G$ generating a subgroup of finite index in G there is a nice natural enumeration of the corresponding Cayley graph $\Gamma_{\langle \bar{g} \rangle}$ then G is virtually abelian.*

By the proof of Proposition 3.3 the group $\mathbf{Z}$ satisfies the assumptions of this conjecture. By Proposition 3.1 for any group G satisfying these assumptions and any field F the group ring FG satisfies the ascending chain condition for left ideals. It is a well-known question if G with this property is virtually polycyclic ([26]). Our further results are motivated by those remarks.

In the rest of the section we concentrate on polycyclic torsion-free groups. Our goal is to show that Theorem 3.5 cannot be generalised in this class. Since a finitely generated polycyclic group is virtually torsion-free, that confirms the conjecture for polycyclic groups. Then by Proposition 3.1 we see that the conjecture holds in the class of solvable groups.

Let G be a polycyclic group. It is well-known (see [20], [24]) that there are $Aut(G)$-invariant subgroups $N \leq M \leq G$, where M is a torsion-free subgroup of G of finite index, N is a maximal nilpotent normal subgroup of M and M/N is a free abelian group (in the terminology of [24], M is a strongly polycyclic group and N is its nilpotent radical). Take the upper central series of N: $N = N_0 > N_1 > \cdots > N_{l+1} = \{1\}$. Then each N_i/N_{i+1} is torsion-free. Note that any N_i is a normal subgroup of G.

The following situation is a natural generalisation of the assumtions of Theorem 3.5. Let $M = G$ and

$$c_1, \ldots, c_k, a_{0,1}, \ldots, a_{0,k_0}, a_{1,1}, \ldots, a_{l,1}, \ldots, a_{l,k_l}$$

be a sequence generating G where $c_1N, \ldots, c_kN$ are free generators of G/N and each tuple $a_{i,1}N_{i+1}, \ldots, a_{i,k_i}N_{i+1}$ freely generates N_i/N_{i+1}. Let Γ_G be the Cayley graph of G with respect to these generators.

THEOREM 3.9: *In the situation above, if G is not virtually abelian then any natural enumeration of Γ_G is not nice.*

The following lemma will be in use throughout the proof.

LEMMA 3.10: *Let the Cayley graph of a group $(L, g_1, \ldots, g_k)$ have a nice natural enumeration and H be a normal subgroup of L. Then the Cayley graph of L/H under the subsequence of $g_1H, \ldots, g_kH$ consisting of non-trivial elements has a nice natural enumeration.*

Proof: We order $\Gamma_{L/H}$ by the representatives of cosets with the least numbers with respect to the enumeration of Γ_L. It is clear that these representatives define the shortest words in the corresponding cosets. Therefore they define the shortest words under $g_1H, \ldots, g_kH$ representing the corresponding cosets. This guarantees that the obtained enumeration is natural.

To see that the enumeration is nice, notice that if w and w' are the chosen representatives of wH and $w'H$ respectively and $g \in L$ takes w onto w' and moves a nice initial segment determined by w into the corresponding segment determined by w' then gH moves the initial segment determined by wH in $\Gamma_{L/H}$ into the corresponding one determined by $w'H$. Now it is easily seen that there is no infinite anti-chain of nice initial segments under the above enumeration of $\Gamma_{L/H}$. ∎

We now consider the nilpotent case: $G = N$.

PROPOSITION 3.11: *If the group N admits a nice natural enumeration of its Cayley graph with respect to a generating sequence as above then N is virtually abelian.*

Proof: Let N be generated by $a_{0,1}, \ldots, a_{l,k_l}$ and a central normal series $N = N_0 > N_1 > \cdots > N_l > \{1\}$ be chosen as above. Suppose that G is not virtually abelian. Then $l \geq 1$. Let s be the minimal number such that there are $a_{0,i} \in N_0$ and $a_{s,j} \in N_s$ with $[a_{0,i}, a_{s,j}] \neq 1$. Denote by g and g' the corresponding elements from N_0 and N_s. By Lemma 3.10 we may assume that $[(g')^{-1}, g^{-1}] \in N_l = C(N)$.

The main point of our proof is to show that for sufficiently large m the length of the shortest word presenting $g'g^m(g')^{-1}g^n$ is $n+m+2$ (where $m, n \in \omega$). Suppose the contrary. Then $g'g^m(g')^{-1}g^n = g^t(a_{i,j})^e g^r$ where $n+m = t+s$

and $e \in \{-1, 0, 1\}$ (note that the shortest presentation of this word in N/N_1 is $g^{m+n}N_1$). Since $[(g')^{-1}, g^{-1}] \in C(N)$ we see that $[(g')^{-1}, g^{-1}]^m = a^e_{i,j}$, $e \neq 0$ and $a_{i,j} \in N_l$. If there are infinitely many m satisfying such a condition then we obtain $[(g')^{-1}, g^{-1}]^m = [(g')^{-1}, g^{-1}]^{m'}$ for some $m \neq m'$, which contradicts the fact that N is torsion-free.

It is obvious that the length of the word $g'g^n$ is $n + 1$. This allows us to finish the proof as follows. Take a sequence $n_0 < n_1 < \cdots$ of natural numbers such that any $n_{i+1} - n_i$ is sufficiently large. Now note that the initial segments determined by $g'g^{n_i}$ form an antichain with respect to the given enumeration of Γ_N. Indeed, an element from N which takes $g'g^{n_i}$ onto $g'g^{n_j}$ (of course we may assume that $i < j$) must move the last element of the form g^m from the initial segment determined by $g'g^{n_i}$, onto $g'g^{n_j - n_i}(g')^{-1}g^m$. Since the enumeration is natural we may assume that for all i the last element of this form is g^{n_i} (the other possibility is g^{n_i+1}). However, this implies that $g'g^{n_j-n_i}(g')^{-1}g^{n_i}$ cannot be enumerated before $g'g^{n_j}$ because its length is $n_j + 2$. We have a contradiction with the existence of a nice natural enumeration. ∎

Proof of Theorem 3.9: Let G be a torsion-free polycyclic group which is not virtually abelian and the generators of G satisfy the conditions of Theorem 3.9 with respect to the upper central series $N = N_0 > \cdots > N_{l+1} = 1$ of its maximal nilpotent normal subgroup N. Assume that Γ_G has a nice natural enumeration. By Proposition 3.11 we have that $N \neq G$.

We now follow the proof of Theorem 4.3 from [28]. The algebraic material that we use below can be found in [24]. Let D be the unique connected simply connected nilpotent Lie group containing N as a discrete subgroup with coset space D/N compact. For $g \in G$, let $\xi(g)$ denote the unique Lie group automorphism of D such that $x \to gxg^{-1}$ for $x \in N$, and $\xi_*(g)$ be the induced automorphism of the Lie algebra $\mathcal{D}$ of D. Thus $exp(\xi_*(g)(v)) = \xi(g)(exp(v))$ (it is convenient to think that all the groups that we consider are subgroups of $GL(n, \mathbf{C})$, [28]). It is shown in Proposition 4.4 of [28] that the group $U = \{g \in G\colon \xi_*(g)$ has every eigenvalue $1\}$ is nilpotent and normal in G and $N \leq U$. Thus $N = U$. If every generator $c_i \in G \setminus N$ defines $\xi_*(c_i)$ having every eigenvalue of absolute value 1 then by the proof of Proposition 4.4 of [28] the corresponding generators of G/N are of finite order. This contradicts our assumptions.

Let D_i be the analytic subgroup of D containing N_i such that D_i/N_i is compact and $\mathcal{D}_i$ be the corresponding Lie algebra. We may now assume that c_1 defines $\xi_*(c_1)$ having an eigenvalue λ not of absolute value 1 which occurs on $\mathcal{D}_r/\mathcal{D}_{r+1}$ and r is minimal with respect to this condition and all c_i. In particular, we have that $\xi(c_1)$ is of infinite order on N_r/N_{r+1}. By Lemma 3.10 we may assume that $r = l$.

The proof of Theorem 4.3 from [28] provides $t \in \mathbf{Z}$ and an element $a \in N_l$ such that the elements of the form

$$\Pi_{-r\leq i\leq s}(c_1^{-it}ac_1^{it})^{\tau_i}, \quad \{r,s\} \subset \omega,\ \tau_i \in \{0,1\},$$

are pairwise distinct. We follow at this place the argument from [27] (which looks slightly easier than that in the original paper). The number t is chosen such that $\xi_*(c_1^t)$ has an eigenvalue λ with $|\lambda| \geq 10$. Let $\{v_1,\ldots,v_{k_l}\}$ be the basis of $\mathcal{D}_l$ such that $exp(v_i) = a_{l,i}$. By Jordan decomposition one can find a non-trivial linear form $\beta : \mathcal{D}_l \to \mathbf{C}$ such that $\beta\cdot\xi_*(c_1^t) = \lambda\beta$. Then for any $v \in \mathcal{D}_l$ such that $\beta(v) \neq 0$ we have

$$\beta(\Sigma\tau_i\xi_*(c_1^t)^i(v)) = (\Sigma\tau_i\lambda^i)\cdot\beta(v).$$

It is clear that the numbers $\Sigma\tau_i\lambda^i$, $\tau_i \in \{0,1\}$, are pairwise distinct. So, the elements $exp(\Sigma\tau_i\xi_*(c_1^t)^i(v))$ are pairwise distinct. Then for $a = exp(v)$ the elements $\Pi(c_1^{-it}ac_1^{it})^{\tau_i}$ also are pairwise distinct. Since v is a linear combination of $v_1,\ldots,v_{k_l}$, we may assume that $v = v_i$, $1 \leq i \leq k_l$. Then $a = a_{l,i}$ is a required element.

Assume that a is the first in $\{a_{l,1},\ldots,a_{l,k_l}\}$. As in the nilpotent case we will show that for sufficiently large $m \in \omega$ the length of the shortest word presenting $ac_1^{mt}a^{-1}c_1^{nt}$ is $nt + mt + 2$.

Suppose the contrary. Then there are $b \in \{a_{l,1},\ldots,a_{l,k_l}\}$ and $e \in \{-1,0,1\}$ such that for infinitely many m we have $ac_1^{mt}a^{-1}c_1^{nt} = c_1^s b^e c_1^r$ where $nt+mt = r+s$ (note that the shortest presentation of this word in G/N is $c_1^{mt+nt}N$). Also note that the equation

$$b^e = c_1^{-s}ac_1^s\cdot c_1^{r-nt}a^{-1}c_1^{nt-r}$$

guarantees that $b \in N_l$ above. As a result we have infinitely many words of the form $c_1^{-s}ac_1^s\cdot c_1^{r-nt}a^{-1}c_1^{nt-r}$ presenting the same element. Conjugating b^e by some c_1^i with $i < t$ we can arrange that $t|\ s$ and $t|\ r$ in these words. Then for some $s < s'$ and $r \neq r'$ with $t|\ s,s'$ and $t|\ r,r'$ we obtain the equation

$$c_1^{-s}ac_1^s\cdot c_1^{r'-nt}ac_1^{nt-r'} = c_1^{-s'}ac_1^{s'}\cdot c_1^{r-nt}ac_1^{nt-r}.$$

This contradicts the choice of t and a.

The rest of the proof is the same as in Proposition 3.11. ∎

4. Finite covers and the finite model property

One of the most known achievements of stability theory is Hrushovski's theorem that a totally categorical theory can be axiomatized by a sentence that has large finite models, together with axioms stating that the structure is infinite [14]. The proof involves covers and nice enumerations. It is natural to ask if in the case of structures of bounded valency similar statements exist.

Notice that the Cayley graph of $\mathbf{Z}^2$ is not axiomatizable by a sentence and the axioms stating that the models of T are infinite. Indeed, any sentence of the theory holds in the Cayley graph of an appropriate $\mathbf{Z}(k) \times \mathbf{Z}$. This suggests that quasifinite axiomatizability should be reformulated as was done in [14] in the case of ω-stable ω-categorical structures.

The standard situation for ω-stable ω-categorical covers can be described as follows.

> Let π: $M \to W$ be a finite cover of W. Then there is a sentence Φ such that (i) $\{\Phi\} \cup Th(W)$ axiomatizes the cover, (ii) if T' is a sufficiently large finite subset of $Th(W)$ and $W' \models T'$, then W' extends to a model of Φ.

Notice that the existence of Φ with properties (i) and (ii) together with the finite model property for W imply the finite model property for M.

On the other hand, if a sentence Φ satisfying (i) and (ii) exists, then any Ψ satisfying (i) satisfies (ii) too (because any two Φ and Φ' satisfying (i) are equivalent with respect to a sufficiently large subset of $Th(W)$). Note that Theorem 2.2 implies that if a structure W of bounded valency has a nice enumeration then property (i) is satisfied for any finite cover of W.

However, the case of ω-stable ω-categorical structures is *extremely lucky*: property (i) is guaranteed by nice enumerability and property (ii) follows from the construction of *envelopes* (see [5]). The latter does not involve nice enumerations and in fact is a strong version of the finite model property. In our case (under the presence of Φ with property (i)) an envelope can be defined as a finite cover $M'' \to W''$ satisfying Φ where W'' is finite and satisfies a sufficiently large finite subset of $Th(W)$. Such a definition ensures some kind of homogeneity resembling that in [5]. Unexpectedly, Theorem 4.11 (see below) claims that in very natural situations the finite model property realized by envelopes implies the existence of an expansion M' of M such that π induces a finite cover $M' \to W$ with finite kernel.

That makes it natural to study the finite model property for superlinked finite covers of Cayley graphs. In the first part of the section we shall show that the finite model property holds for any finite superlinked cover of the Cayley graph of a finitely generated virtually nilpotent group.

On the other hand, we give an example of a cover with a nice enumeration but without covering expansions with finite kernel. It is unknown if this can happen for finite covers of Cayley graphs of abelian groups. The author does not know if these covers have the finite model property. It will be only shown that any finite cover of the Cayley graph of $\mathbf{Z}^2$ has the finite model property.

We close this introduction by an example showing that *property (ii) in the formulation above does not hold even for finite superlinked covers of the Cayley graph of* $\mathbf{Z}^2$ (but, as we have already mentioned, this does not destroy the situation too much).

Example: Let $U_2 = (\mathbf{Z}^2, u_1, u_2)$ be the algebra of two unary functions realizing the Cayley graph of $\mathbf{Z}^2$ with respect to the standard free generators. Let $M = \mathbf{Z}^2 \times \{0,1\}$ be a finite cover of U_2 with respect to the natural projection π. The structure M consists of the structure U_2, the projection π and unary 1-1-functions s_1 and s_2 defined as follows.

Let $s_1((z,z',i)) = (z+1,z',i)$, $i \le 1$. If z is even let $s_2((z,z',0)) = (z,z'+1,0)$ and $s_2((z,z',1)) = (z,z'+1,1)$. If z is odd let $s_2((z,z',0)) = (z,z'+1,1)$ and $s_2((z,z',1)) = (z,z'+1,0)$.

As a result the structure M satisfies the following sentence Φ:

$$\forall x(s_1^2 s_2(x) = s_2 s_1^2(x) \wedge s_2^2 s_1(x) = s_1 s_2^2(x) \wedge s_1 s_2(x) \neq s_2 s_1(x) \wedge$$
$$s_1 s_2 s_1(x) \neq s_2 s_1^2(x) \wedge s_2 s_1 s_2(x) \neq s_1 s_2^2(x) \wedge \pi(s_i(x)) = u_i(\pi(x))).$$

It is easy to see that $Aut(M)$ acts transitively on M. The kernel of the cover $M \to U_2$ consists of two elements.

We claim that *for no odd k is there a cover of the Cayley graph of $\mathbf{Z}(k) \times \mathbf{Z}$ (with respect to functions u_1 and u_2 with $u_1^k(x) = x$) in the language of M and satisfying Φ.*

Assume that such a cover exists. The elements of the cover can be considered as triples (z,z',i) with $z \in \mathbf{Z}(p)$, $z' \in \mathbf{Z}$ and $i \in \{0,1\}$. Now by the last equality in Φ, $s_1^k((0,0,0)) = (0,0,0)$ or $s_1^k((0,0,0)) = (0,0,1)$. In the first case the preimage of $\{(0,0),\ldots,(k-1,0)\} \subset U_2$ with respect to π consists of two s_1-cycles, say $\{(0,0,0),\ldots,(k-1,0,0)\}$ and $\{(0,0,1),\ldots,(k-1,0,1)\}$. On the other hand,

$$s_1^k s_2((0,0,0)) = s_1 s_2 s_1^{k-1}((0,0,0)) \neq s_2 s_1^k((0,0,0)) = s_2((0,0,0)).$$

This means that the preimage of $\{(0,1),\ldots,(k-1,1)\}$ with respect to π consists of one s_1-cycle. Now the automorphism of U_2 taking $(0,0)$ to $(0,1)$ cannot be extended to an automorphism of M.

If $s_1^k((0,0,0)) = (0,0,1)$, the preimage of $\{(0,0),\ldots,(k-1,0)\} \subset U_2$ with respect to π consists of one s_1-cycle. On the other hand,

$$s_1^k s_2((0,0,0)) = s_1 s_2 s_1^{k-1}((0,0,0)) \neq s_2 s_1^k((0,0,0)) = s_2((0,0,1)).$$

This means that the preimage of $\{(0,1),\ldots,(k-1,1)\}$ with respect to π consists of two s_1-cycles, say $\{(0,1,0),\ldots,(k-1,1,0)\}$ and $\{(0,1,1),\ldots,(k-1,1,1)\}$. Then the automorphism of U_2 taking $(0,0)$ to $(0,1)$ cannot be extended to an automorphism of M.

The same argument shows that *for no odd k is there a cover of the Cayley graph of $\mathbf{Z} \times \mathbf{Z}(k)$ (with respect to functions u_1 and u_2 with $u_2^k(x) = x$) in the language of M and satisfying Φ.*

We now claim that *for no odd k and l is there a model based on the Cayley graph of $\mathbf{Z}(k) \times \mathbf{Z}(l)$ (with $u_1^k(x) = x$ and $u_2^l(x) = x$) in the language of M and satisfying Φ.*

Assume that such a model exists. Its elements can be considered as triples (z, z', i) with $z \in \mathbf{Z}(k)$, $z' \in \mathbf{Z}(l)$ and $i \in \{0,1\}$. Now by the last equality in Φ, $s_1^k((0,0,0)) = (0,0,0)$ or $s_1^k((0,0,0)) = (0,0,1)$. In the first case, for all even $m < l$ the preimage of $\{(0,m),\ldots,(k-1,m)\} \subset U_2$ with respect to π consists of two s_1-cycles $\{(0,m,0),\ldots,(k-1,m,0)\}$ and $\{(0,m,1),\ldots,(k-1,m,1)\}$ and the preimage of $\{(0,m+1),\ldots,(k-1,m+1)\}$ with respect to π consists of one s_1-cycle.

If $s_1^k((0,0,0)) = (0,0,1)$, for all even $m < l$ the preimage of $\{(0,m),\ldots,(k-1,m)\} \subset U_2$ with respect to π consists of one s_1-cycle and the preimage of $\{(0,m+1),\ldots,(k-1,m+1)\}$ with respect to π consists of two s_1-cycles: $\{(0,m+1,0),\ldots,(k-1,m+1,0)\}$ and $\{(0,m+1,1),\ldots,(k-1,m+1,1)\}$.

We now have a contradiction because l is odd and $l = 0$ in $\mathbf{Z}_l$. ∎

4.1. The finite model property for Cayley graphs. The main result of this subsection describes when a structure with finite point stabilizer has the finite model property. Then we show that superlinked (with finite kernel) finite covers of Cayley graphs of virtually nilpotent groups have the finite model property.

We start with a construction of a quotient of a structure of bounded valency.

Quotients of structures of bounded valency. Let W be a structure of bounded valency and $G \leq Aut(W)$ have a finite number of orbits on W and all point-stabilisers be trivial. Let K be a normal subgroup of G such that no orbit of K contains a pair a, b with $d(a,b) \leq 2m+1$. We now define W/K as follows.

Define an equivalence relation θ on W by:

$$(v_1, v_2) \in \theta \leftrightarrow \exists \gamma \in K(\gamma(v_1) = v_2).$$

Then W/θ becomes a structure of the language of W with respect to the relations obtained from the relations R of W as follows:

$$(v_1\theta, \ldots, v_k\theta) \in R^\theta \leftrightarrow (\exists v'_1 \in v_1\theta, \ldots, v'_k \in v_k\theta)(W \models R(v'_1, \ldots, v'_j, \ldots, v'_k)).$$

By the choice of K, if $(v_1, v_2) \in \theta$ and $d(v_1, v_2) < 2m + 1$ then $v_1 = v_2$. This implies that the m-ball of any v in W maps injectively into the m-ball of $v\theta$ in W/θ.

On the other hand, if $(v_1\theta, \ldots, v_k\theta) \in R^\theta$ then there are $\alpha_1, \ldots, \alpha_k \in K$ such that $W \models R(\alpha_1(v_1), \ldots, \alpha_k(v_k))$. Then

$$W \models R(v_1, \alpha_1^{-1}\alpha_2(v_2), \ldots \alpha_1^{-1}\alpha_k(v_k)).$$

This shows that the map $w \to w\theta$ is surjective between the sets of neighbours of any v and $v\theta$ respectively. This in turn implies that the m-ball of v is isomorphic to the m-ball of $v\theta$. ∎

Notice that in this construction the group G is finitely generated: let l be the minimal distance between the elements of $Gv \subseteq W$; then the set of automorphisms taking v onto each of the points from Gv at distance l serves as a tuple of generators. Let $\{g_1, \ldots, g_n\}$ be such a set of generators.

Now choose $v_1, \ldots, v_s$, representatives of all the G-orbits. Then for every m it is easy to find a number k such that for every v_i, if a word α in $\{g_1, \ldots, g_n\}$ takes v_i into the $(2m+1)$-ball of v_i, then $ln(\alpha) \leq k$ (because the point-stabilizers are trivial).

Lemma 4.1: *Let H be a normal subgroup of G such that H does not contain any word in $\{g_1, \ldots, g_n\}$ of length $\leq k$, presenting a non-trivial element of G. Then no orbit of H contains a pair a, b with $d(a, b) \leq 2m + 1$.*

Proof: If the lemma does not hold for a and b, then by normality we may assume that a coincides with the corresponding v_i. By triviality of stabilizers the only element of H taking v_i to b is presented by a word of length $\leq k$. This is a contradiction. ∎

Now assume that the permutation structure (G, W) is transitive. Then all g_i take $v = v_1$ to its neighbours. Let $\psi: G_1 \to G$ be a homomorphism, where $G_1 = \langle g'_1, \ldots, g'_n \rangle$, $\psi(g'_i) = g_i$ and $K' = Ker\psi$ does not contain any word in

$\{g'_1, \ldots, g'_n\}$ of length $\leq 2m+3$. Define a structure of the language of W on G_1 as follows:

$$(\alpha'_1, \ldots, \alpha'_k) \in R \leftrightarrow (\exists j \leq k)(\exists g'_{i_1}, \ldots, g'_{i_{j-1}}, id, \ldots, g'_{i_k} \in G_1)$$
$$\bigwedge_{j \neq r \leq k} \alpha'_r = \alpha'_j g'_{i_r} \wedge W \models R(g_{i_1}(v), \ldots, v, \ldots, g_{i_k}(v))).$$

Notice that the relations are invariant with respect to the action of G_1 by the left multiplication.

LEMMA 4.2: *The structures G_1 and W have the same m-balls of a point. In particular, if for some $\alpha'_1, \ldots, \alpha'_k, \beta \in G_1$ any $d(\alpha'_i, \beta)$, $1 \leq i \leq k$, is not greater than m and $\alpha_i = \psi(\alpha'_i)$, then*

$$W \models R(\alpha_1(v), \ldots, \alpha_k(v)) \leftrightarrow G_1 \models R(\alpha'_1, \ldots, \alpha'_k).$$

Proof: We start with the particular case of the lemma. The direction

$$G_1 \models R(\alpha'_1, \ldots, \alpha'_k) \to W \models R(\alpha_1(v), \ldots, \alpha_k(v))$$

follows from the definition of the structure on G_1: since

$$W \models R(g_{i_1}(v), \ldots, v, \ldots, g_{i_k}(v)) \quad \text{and} \quad \bigwedge_{j \neq r \leq k} \alpha'_r = \alpha'_j g'_{i_r},$$

then applying the automorphism $\alpha_j = \psi(\alpha'_j)$ we see

$$W \models R(\alpha_1(v), \ldots, \alpha_j(v), \ldots, \alpha_k(v)).$$

To see the converse it suffices to prove that

$$G_1 \models R((\alpha'_j)^{-1}\alpha'_1, \ldots, (\alpha'_j)^{-1}\alpha'_k)).$$

Since $W \models R(\alpha_j^{-1}\alpha_1(v), \ldots, \alpha_j^{-1}\alpha_k(v))$, there is a sequence $g_{i_1}, \ldots, id, \ldots, g_{i_k}$ (satisfying $W \models R(g_{i_1}(v), \ldots, v, \ldots, g_{i_k}(v)))$ such that $\alpha_j^{-1}\alpha_s = g_{i_s}$, $j \neq s \leq k$.

On the other hand, by the definition of the structure on G_1, there are $t_{1,1}, \ldots, t_{1,p}, \ldots, t_{k,1}, \ldots, t_{k,q}$ such that $p \leq 2m, \ldots, q \leq 2m$, and

$$(\alpha'_j)^{-1}\alpha'_1 = g'_{t_{1,1}} \cdots g'_{t_{1,p}}, \ldots, (\alpha'_j)^{-1}\alpha'_j = id, \ldots, (\alpha'_j)^{-1}\alpha'_k = g'_{t_{k,1}} \cdots g'_{t_{k,q}}.$$

We now see that

$$(g'_{i_r})^{-1} g'_{t_{r,1}} \cdots g'_{t_{r,l}} K' = (g'_{i_r})^{-1}(\alpha'_j)^{-1}\alpha'_r K' = (g'_{i_r})^{-1} g'_{i_r} K' = K'.$$

By the choice of K' we obtain $g'_{t_{r,1}} \cdots g'_{t_{r,l}} = g'_{i_r}$, $r \leq k$. This shows that

$$G_1 \models R((\alpha'_j)^{-1}\alpha'_1, \ldots, (\alpha'_j)^{-1}\alpha'_k)).$$

We now see that the homomorphism $G_1 \to G$ embeds the m-ball of id into the m-ball of v (the injectivity follows by a similar argument). The definition of R on G_1 shows that the embedding is surjective for 1-balls. This implies that W and G_1 have the same m-balls. ∎

We are ready to state the main result of the subsection.

THEOREM 4.3: *Let W be a strongly minimal, connected structure of bounded valency. Let $G \leq Aut(W)$ be a finitely generated group acting transitively on W with finite point stabilizer.*

The structure W has the finite model property if and only if, for any tuple of generators $g_1, \ldots, g_n$ of G and any finite sets Σ_1 and Σ_2 of words in $\{g_1, \ldots, g_n\}$ with $\forall \alpha \in \Sigma_1 (G \models \alpha = 1)$ and $\forall \alpha \in \Sigma_2 (G \models \alpha \neq 1)$, there is a normal subgroup of finite index of the group $\langle g_1, \ldots, g_n; \Sigma_1 \rangle$ which does not intersect Σ_2.

Proof: Let W have the finite model property and $v \in W$. Let t be chosen so that W does not have non-trivial automorphisms fixing $B(t, v)$ pointwise. Then for the type p of some enumeration of $B(t, v)$ the group G acts on the structure W^p with trivial point stabiliser (see Section 1.3). On the other hand, by Lemma 1.5 the structure W^p has the finite model property.

For $g_1, \ldots, g_n$ and finite sets Σ_1 and Σ_2 as in the statement of the theorem let

$$m = max(length(\alpha)\colon \alpha \in \Sigma_1 \cup \Sigma_2) \cdot max(d(v, g_i(v))\colon i \leq n) + 1.$$

Let U^p be a finite structure with the same m-balls as in W^p. Then for any $u \in U^p$ define $g_i(u)$ as u' such that the type of the pair (u, u') in $B(m, u)$ coincides with the type of $(v, g_i(v))$ in $B(m, v)$. (Notice that by the definition of I for any $\hat{v}'$ adjacent with $\hat{v}$ with respect to the relation I there is a unique number i such that $\hat{v}'$ occurs as $\hat{v}_i$ in the formula $I(\hat{v}, \hat{v}_1, \ldots, \hat{v}_k)$ holding in W^p.) This defines a tuple of permutations on U^p satisfying the relators from Σ_1. Thus we have an action of $\langle g_1, \ldots, g_n; \Sigma_1 \rangle$ on U^p. Let K be the kernel of the action. By the choice of m, the group K does not meet Σ_2. Since U^p is finite, K is of finite index.

To prove the converse of the theorem choose t and W^p as above. By Lemma 1.5 it suffices to prove that W^p has the finite model property. Let $g_1, \ldots, g_n$ be the set of all automorphisms taking $\hat{v}$ to adjacent points. Let $m \in \omega$ and Σ_1 be the set of all words in $\{g_1, \ldots, g_n\}$ of length $\leq 2m + 3$ presenting identity under the action of G on W^p. Let $G_1 = \langle g_1', \ldots, g_n'; \Sigma_1 \rangle$. Using the action of G_1 on W^p define a structure on G_1 of the language of the structure W^p as follows:

$$(\alpha_1', \ldots, \alpha_k') \in R \leftrightarrow (\exists j \leq k)(\exists g_{i_1}', \ldots, g_{i_j}', id, \ldots, g_{i_k}')(\bigwedge_{j \neq r \leq k} \alpha_r' = \alpha_j' g_{i_r}' \wedge$$
$$W^p \models R(g_{i_1}(\hat{v}), \ldots, v, \ldots, g_{i_k}(\hat{v}))).$$

Let K be a normal subgroup of G_1 of finite index such that K does not include any non-trivial element which is presented by a word of length $\leq 2m+3$. Then Lemma 4.2 and the construction of quotients together with Lemma 4.1 are applicable to G_1, K and the structure defined on G_1/K. ∎

Remark 4.4: J. G. Thomson and G. Higman have found infinite finitely presented simple groups (see [13]). By our theorem the Cayley graph of such a group does not have the finite model property.

Remark 4.5: Let W be a transitive connected structure of bounded valency having finite point stabilizer and a (natural) nice enumeration. *Does W have the finite model property ?*

In the case of Cayley graphs this provides an interesting group-theoretic question related to the conjecture of Section 3: *Is there a finitely presented group G such that its Cayley graph has a (natural) nice enumeration and G is not residually finite: there is a finite set $A \subset G$ such that any subgroup of finite index intersects A ?* It looks very likely that an inspection of typical Olshanski's examples (rather auxiliary finitely presented groups in these examples) can help to understand the obstacles arising here.

We now apply quotients to covers.

PROPOSITION 4.6: *Let $C \to W$ be a finite cover of the Cayley graph of a finitely generated, virtually nilpotent group. If the kernel of C is finite then C has the finite model property.*

Proof: Since $Aut(W)$ acts regularly on W, the stabilizer $Aut(C/c)$ of a point $c \in C$ is finite. Taking a finite cover of C (as in Theorem 4.3) we can arrange that the stabilizer is trivial. It is clear that $Aut(C)$ is finite-by-nilpotent. By a theorem of P. Hall the group is virtually nilpotent. On the other hand, finitely generated virtually nilpotent groups are residually finite ([20], [24]).

Let $m \in \omega$ and $v \in C$ be such that $B(m+1, v)$ meets all $Aut(C)$-orbits. Take $K \triangleleft Aut(C)$ such that no pair from $B(3m+3, v)$ is in the same K-orbit. Then there is no pair $a, b \in C$ with $d(a,b) \leq 2m+1$ and of the same K-orbit.

Applying the construction of quotients we obtain a finite structure C/θ with the same types of m-balls as in C. This proves the proposition. ∎

4.2. THE FINITE MODEL PROPERTY FOR FINITE COVERS.

1. Finite covers of Cayley graphs of abelian groups. Let W be the Cayley graph of a finitely generated virtually abelian group G and let A be an abelian torsion free normal subgroup of G of finite index. Then A is isomorphic to $\mathbf{Z}^n$ for some n. Then w.l.o.g. we may assume that W is the Cayley graph of G considered with respect to a tuple of generators $z_1, \ldots, z_n, h_1, \ldots, h_k$ where $z_1, \ldots, z_n$ form a tuple of free generators of $\mathbf{Z}^n$ and $h_1, \ldots, h_k$ represent the non-trivial cosets of G/A (all Cayley graphs of G are interdefinable). For each h_i the correspondence $w \to h_i w, w \in A$, defines a copy of the Cayley graph of A which is invariant with respect to A. Note that the action of A on $h_i\Gamma_A$ (by left multiplication) induces the following action of A on Γ_A: $h_i^{-1}ah_i \cdot v$, $v \in \Gamma_A$. It is easily seen that this action is transitive. In particular, we have that the graph on $h_i\Gamma$ induced by the generators $h_i^{-1}z_jh_i$, $1 \leq j \leq n$, is interdefinable with the graph induced by $z_1, \ldots, z_n$.

LEMMA 4.7: *For any natural number s there exists a natural number t such that if $a, b \in W$ of the same copy of Γ_A are distant at $\geq t$ with respect to $z_1, \ldots, z_n$ then the distance $d(a, b)$ in W is at least s.*

Proof: If the statement does not hold, then by transitivity there is a point $a \in W$ having infinitely many neighbours. This contradicts that W is of finite valency. ∎

LEMMA 4.8: *Let $\pi: M \to W$ be a finite cover of W (M is a structure of bounded valency) and W satisfy the assumptions above. For any natural number s there exists a natural number t such that if for $a, b \in M$ the elements $\pi(a)$ and $\pi(b)$ are of the same copy of Γ_A and are distant at $\geq t$ with respect to $z_1, \ldots, z_n$ then the distance $d(a, b)$ in M is at least s.*

Proof: The argument of the previous lemma works in this case. ∎

PROPOSITION 4.9: *Let a group G be a finitely generated virtually abelian group. Let W be the Cayley graph of G and $\pi: M \to W$ be a finite cover of W. Then $Th(M)$ is not finitely axiomatizable.*

Proof: We preserve the assumptions of the previous lemmas. We also assume that if $\pi(a)$ and $\pi(b)$ are the same or adjacent in W then the elements a and b

are adjacent in M. We want to show that if a sentence ϕ holds in M then there is a structure M' satisfying ϕ but not elementarily equivalent to M.

Let $M \models \phi$. We may assume that the sentence ϕ describes the s-balls of M for some s. Let $\alpha \in Aut(M)$ be an automorphism inducing the corresponding generator z_n. By the lemmas above there exists t_n such that for any $v \in M$ and $t \geq t_n$, $d(v, \alpha^t(v)) \geq 2s+1$ with respect to M. Define an equivalence relation θ on M as follows:

$$(v_1, v_2) \in \theta \leftrightarrow \exists m \in \mathbf{Z}(t_n |\, m \wedge \alpha^m(v_1) = v_2).$$

Then M/θ becomes a structure of the language of M with respect to the relations obtained from the relations R of M as follows:

$$(v_1\theta, \ldots, v_k\theta) \in R^\theta \leftrightarrow \exists v_1' \in v_1\theta, \ldots, v_k' \in v_k\theta R(v_1', \ldots, v_k').$$

Notice that by the choice of α, if $(v_1, v_2) \in \theta$ and $d(v_1, v_2) < 2s+1$ then $v_1 = v_2$. This implies that the s-ball of v in M maps injectively into the s-ball of v in M/θ.

On the other hand, if $(v_1\theta, \ldots, v_k\theta) \in R^\theta$ then there are $m_1, \ldots, m_k \in \mathbf{Z}$ such that $M \models R(v_1', \ldots, v_k')$, where $v_1' = \alpha^{m_1}(v_1), \ldots, v_k' = \alpha^{m_k}(v_k)$. Then

$$M \models R(v_1, \alpha^{m_2 - m_1}(v_2), \ldots \alpha^{m_k - m_1}(v_k)).$$

This shows that the map $w \to w\theta$ is surjective between the sets of neighbours of v and $v\theta$ respectively. This in turn implies that the s-ball of v is isomorphic to the s-ball of $v\theta$. Then $M/\theta \models \phi$.

Let $v \in \Gamma_A \subset W$. Then the the map $w \to w\theta$ is not injective on the (t_n+1)-ball of v. It is easy to see that the size of the (t_n+1)-ball of v/θ is less than the size of the (t_n+1)-ball of v. ∎

We now prove the finite model property for finite covers of the Cayley graph W in the case when the subgroup A is isomorphic to $\mathbf{Z}^2$. The general case is open.

PROPOSITION 4.10: *Let a group G be a finitely generated virtually abelian group and let $A \cong \mathbf{Z}^2$ be a normal subgroup of G of finite index. Let W be the Cayley graph of G and $\pi: M \to W$ be a finite cover of W. Then $Th(M)$ has the finite model property.*

Proof: We preserve the assumptions above. We want to show that if a sentence ϕ holds in M then there is a finite structure M' satisfying ϕ.

Let $M \models \phi$. We may assume that the sentence ϕ describes s-balls of M for some s which is greater than $3 = ln(h_i^{-1} z_j h_i)$, $i \leq k, 1 \leq j \leq 2$. Let $\alpha_1, \alpha_2 \in Aut(M)$ be automorphisms inducing the generators z_1, z_2 of A. By the lemmas above there exists t_i such that for any $v \in M$ and $t \geq t_i$, $d(v, \alpha_i^t(v)) \geq 4s + 1$ with respect to M.

Take $v_0 \in M$ with $\pi(v_0) \in \Gamma_A$. Let $v_1, \ldots, v_k$ be chosen so that $\pi(v_i)$, $i \leq k$, are adjacent to $\pi(v_0)$ and represent all the cosets of G/A. Define

$$D_1 = \{v \in M\colon (\exists i \leq t_1)(\exists j \leq k)(\exists w \in B(s, v_j))(v = \alpha_1^i(w))\}.$$

By our assumptions on s, for every $w \in B(s, v_j)$ the set $\pi(D_1) \cap h_j \Gamma_A$ contains a connected subgraph of $h_i \Gamma_A$ having $\pi(w), \pi(\alpha_1(w)), \ldots, \pi(\alpha_1^{t_1}(w))$ among its vertices.

We consider sets $\alpha_2^l(D_1)$ with respect to the language of M extended by the restriction of α_1 (considered as a partial function). By the pigeon hole principle there are $i_2 < j_2$ such that $t_2 < j_2 - i_2$ and $\alpha_2^{j_2 - i_2}$ is an isomorphism between the structures $\alpha_2^{i_2}(D_1)$ and $\alpha_2^{j_2}(D_1)$. Since the restrictions of α_1 and α_2 on W are commutative, the set $\alpha_2^{i_2}(D_1)$ consists of 'fibres' $\{\alpha_1^l(w)\colon 0 \leq l \leq t_1\}$, $w \in \alpha_2^{i_2}(B(s, v_j))$, $j \leq k$. Let

$$D_2 = \{v \in M\colon (\exists i_2 \leq t \leq j_2)(\exists w \in D_1)(v = \alpha_2^t(w))\}.$$

Notice that every α_i acts on D_2 as a partial isomorphism. As above, for any $w \in D_1$ with $\pi(w) \in h_i \Gamma_A$ the set $\pi(\alpha_2^t(w))$, $i_2 \leq t \leq j_2$, is contained in a connected subgraph of $\pi(D_2) \cap h_i \Gamma_A$.

To construct a finite model satisfying ϕ we apply the following procedure. Identify the elements of $\alpha_2^{i_2}(D_1)$ with their images in $\alpha_2^{j_2}(D_1)$ under the isomorphism $\alpha_2^{j_2 - i_2}$. By D_2' we denote the obtained structure and by D_1' the image of $\alpha_2^{i_2}(D_1)$ in D_2'. Note that the definition of the function α_1 on D_1' is correct because it is preserved under the isomorphism above. Since the functions α_1, α_2 induce free abelian action on every component $h_i \Gamma_A$, the structures $\pi(D_2')$ and D_2' do not contain any cycle of the form $v, \alpha_1(v), \ldots, \alpha_1^k(v)$. On the other hand, it is clear that $\alpha_2^{j_2 - i_2}(v) = v$ for all $v \in D_1'$. By the choice of t_2 there is no s-ball $B' \subset D_1'$ containing a pair of the same α_2-cycle. This implies that such a ball is isomorphic to a ball from D_1.

Notice that $\alpha_1^{t_1}$ induces an isomorphism in D_2' between the structure of the elements of D_2' corresponding to $\alpha_2^l(\bigcup_{j \leq k} B(s, v_j))$, $i_2 \leq l \leq j_2$, and the structure corresponding to $\alpha_2^l \alpha_1^{t_1}(\bigcup_{j \leq k} B(s, v_j))$. This follows from the fact that $\alpha_1^{t_1}$ and $\alpha_2^{j_2 - i_2}$ commute on $\alpha_2^{i_2}(\bigcup_{j \leq k} B(s, v_j))$ in D_2. At the second step we identify the elements of D_2' corresponding to $\alpha_2^l(\bigcup_{j\leq k} B(s,v_j))$, $i_2 \leq l \leq j_2$, with their images in $\alpha_2^l\alpha_1^{t_1}(\bigcup_{j\leq k} B(s,v_j))$ under the isomorphism $\alpha_1^{t_1}$. By the choice of t_1, in the obtained D_2'' there is no s-ball B' containing a pair of the same α_1-cycle. This implies that such a ball is isomorphic to a ball from D_2'. As a result we have a structure D_2'' where each element is contained in an α_1-cycle of the length t_1 and in a α_2-cycle of the length $j_2 - i_2$. Since the s-balls are preserved at every step of the procedure, the structure D_2'' satisfies ϕ. ∎

2. Envelopes and the finite model property. Let W be the Cayley graph of $\mathbf{Z}^n$ with respect to its free generators. The structure W has the finite model property because any $\phi \in Th(W)$ holds in the Cayley graph of some $\mathbf{Z}(m_1)\times\cdots\times\mathbf{Z}(m_n)$.

Let $\pi\colon M \to W$ be a finite cover of W. *Does π satisfy the finite model property in the following strong sense: any $\phi \in Th(M)$ holds in a finite cover of the Cayley graph of $\mathbf{Z}(m_1)\times\cdots\times\mathbf{Z}(m_n)$?* Notice that the finite structures obtained in this way naturally correspond to *envelopes* in totally categorical structures [5].

THEOREM 4.11: *Let $\pi\colon M \to W$ satisfy the assumptions above. If M satisfies the version of the finite model property as above then the structure M has an expansion covering W with finite kernel with respect to π.*

Proof: By Theorem 2.2 there is a sentence ϕ such that $Th(M)$ is axiomatized by $Th(W)\cup\{\phi\}$. We may assume that ϕ describes the s-balls of M for some s. Find $\mathbf{Z}(m_1)\times\cdots\times\mathbf{Z}(m_n)$ such that ϕ is satisfied in a finite cover M' of the corresponding Cayley graph W'. We assume that the numbers m_i are large enough and the 1-ball of a point $a \in M'$ does not contain any b such that $d(\pi(a),\pi(b)) \geq m_i/3$. Let $\alpha_1,\ldots,\alpha_n$ be automorphisms of M' extending the natural generators of $\mathbf{Z}(m_1)\times\cdots\times\mathbf{Z}(m_n)$. Let $v \in W'$ and

$$P_1 = \{\alpha_2^{k_2}\ldots\alpha_n^{k_n}(v)\colon k_i \leq m_i\}.$$

Then the structure M' is the disjoint union of $\pi^{-1}(\alpha_1^i(P_1))$, $i < m_1$. For any $l \in \mathbf{Z}$ we introduce a sequence $M_{l,0},\ldots,M_{l,m_1-1}$ of copies of $\pi^{-1}(\alpha_1^i(P_1))$, $i < m_1$ (denoted by $M_{0,0},\ldots,M_{0,m_1-1}$). Let $M_1 = \bigcup_{l\in\mathbf{Z}}(M_{l,0}\cup\cdots\cup M_{l,m_1-1})$. The relations on M_1 are the natural preimages of the relations of M'; for example, a tuple

$$\bar{a} \subset M_{0,s}\cup M_{0,s+1}\cup\cdots\cup M_{0,0}\cup\cdots\cup M_{0,t}$$

with $t + m_1 - s < m_1/3$, and satisfying a relation R of M', defines the sequence of copies

$$\bar{a}_l \subset M_{l,s}\cup M_{l,s+1}\cup\cdots\cup M_{l+1,0}\cup\cdots\cup M_{l+1,t}$$

satisfying R in M_1. It is clear that M_1 and M' have the same s-balls, $M_1 \models \phi$. On the other hand, the structure M_1 is a finite cover of the Cayley graph of $\mathbf{Z} \times \mathbf{Z}(m_2) \times \cdots \times \mathbf{Z}(m_n)$. Indeed, to extend α_i, $i \geq 2$, to M_1 define $\delta_i \in Aut(M_1)$ by the copies of α_i on all $M_{l,0} \cup \cdots \cup M_{l,m_1-1}$. On the other hand, the automorphism α_1 defines $\delta_1 \in Aut(M_1)$ naturally extending the generator of infinite order (call it again α_1) in $\mathbf{Z} \times \mathbf{Z}(m_2) \times \cdots \times \mathbf{Z}(m_n)$. Indeed, on every set $M_{l,0} \cup \cdots \cup M_{l,m_1-2}$ it is defined by the corresponding copy of α_1 and it is defined on M_{l,m_1-1} by the map $M_{l,m_1-1} \to M_{l+1,0}$ copying the restriction of α_1 from M_{0,m_1-1} to $M_{0,0}$. It is easy to see that $\delta_1^{m_1}$ centralizes any δ_i, $i \geq 2$, defined in such a way.

At the second step find $v \in \pi(M_1)$ and define

$$P_2 = \{\alpha_1^{k_1}\alpha_3^{k_3}\cdots\alpha_n^{k_n}(v): k_1 \in \mathbf{Z}, 0 \leq k_i < m_i, i > 1\}.$$

Then the structure M_1 is the disjoint union of $\pi^{-1}(\alpha_2^i(P_2))$, $1 \leq i < m_2$. As above, for $l \in \mathbf{Z}$ we introduce a sequence $N_{l,0}, \ldots, N_{l,m_2-1}$ of copies of $\pi^{-1}(\alpha_2^i(P_2))$, $i < m_2$. Let $M_2 = \bigcup_{l\in\mathbf{Z}}(N_{l,0} \cup \cdots \cup N_{0,m_2-1})$. Defining relations in an appropriate way we obtain that M_2 covers the Cayley graph of $\mathbf{Z}^2 \times \mathbf{Z}(m_3) \times \cdots \times \mathbf{Z}(m_n)$ and satisfies ϕ. As above, one can find extensions $\delta_1, \ldots, \delta_n$ of $\alpha_1, \ldots, \alpha_n$ so that $\delta_1^{m_1}$ and $\delta_2^{m_2}$ centralize every δ_i.

Continuating this procedure we eventually obtain a cover $M_n \to W$ (by our assumptions W is the Cayley graph of $\mathbf{Z}^n$) which satisfies ϕ. Since $Th(M)$ is axiomatized by $Th(W) \cup \{\phi\}$, the structures M_n and M are isomorphic. We also have a tuple $\delta_1, \ldots, \delta_n$ of extensions of the free generators of $\mathbf{Z}^n$, where every $\delta_i^{m_i}$ centralizes all δ_i. Thus expanding M_n by all $\delta_i^{m_i}$ we still obtain a cover of W. The kernel of that cover is finite, because its elements are determined by the restrictions on the set of fibres corresponding to $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}(v): k_i \leq m_i\}$. ∎

Remark 4.12: It is worth noting that *if a structure W of bounded valency has the finite model property and $\pi: M \to W$ is a finite cover admitting a covering expansion with trivial kernel (π **splits**), then M has the finite model property too.* Indeed, take a covering expansion of $\pi: M \to W$ with trivial kernel. We obtain an action of $Aut(W)$ on M with trivial stabilizer. The finite model property for W naturally implies the finite model property of the covering expansion. By taking reducts we have the conclusion.

The remark can be extended to some covers with finite kernels (see Subsection 4.1). This and Theorem 4.11 suggest the following question. Let $\pi: M \to W$ be a finite cover of W (M also is a structure of bounded valency). Assume that

W has a (natural) nice enumeration and satisfies the finite model property. *Is there a covering expansion of M having a finite kernel ?* As we noted above the question is open if W is the Cayley graphs of a finitely generated abelian group.

We now give an example which shows that in general the question has the negative answer.

Example: The structure W is a finite (binary) cover of $U_1 = (\mathbf{Z}, u_1)$, the algebra of a unary function realizing the Cayley graph of $\mathbf{Z}$ with respect to the standard free generator. Let $W = \mathbf{Z} \times \{0,1\}$ be the principal cover of U_1 (there are no new relations) and let $M = \mathbf{Z} \times \{0,1\} \times \{0,1,2,3\}$ be a finite cover of W with respect to the natural projection π (removing the last coordinate). The structure W is considered with respect to the fibre equivalence relation and the relation $\hat{s}_1((x,*),(x',*)) \leftrightarrow u_1(x) = x'$. Notice that $Aut(W)$ is topologically generated by the free generator of $\mathbf{Z}$ (with the action preserving the natural transversals) and any switching s_x: $(x,0) \to (x,1)$ (fixing all points of fibres distinct from x). Since W is a finite cover of the Cayley graph of an abelian group, the structure W has a nice enumeration.

The structure M consists of the structure W, the projection π, binary relations C and S_1 and the unary 1-1-function

$$\gamma\colon (x,0,0) \to (x,1,0) \to (x,0,1) \to (x,1,1) \to (x,0,2) \to (x,1,2) \to$$

$$(x,0,3) \to (x,1,3) \to (x,0,0).$$

The relation C is the equivalence relation with the classes of the form: $(z,0,i)$, $(z,1,i)$, $z \in \mathbf{Z}$, $i < 4$. The relation S_1 is defined as follows. For any z and z' let $((z,i,j),(z+1,i,j')) \in S_1$ iff $j'-j$ is even and $((z,i,j),(z+1,1-i,j')) \in S_1$ iff $j'-j$ is odd.

Let δ be the automorphism M defined by $(z,i,j) \to (z+1,i,j)$. For any $\alpha \in Aut(W)$ there exists $m \in \mathbf{Z}$ such that α and δ^m agree on the first coordinates. Let $A_\alpha = \{z \in \mathbf{Z}\colon \alpha \text{ and } \delta^m \text{ do not agree on the second coordinate of } (z,0)\}$. It is clear that α can be extended to an automorphism of M by $\alpha(z,i,j) = (\alpha(z,i), j+1)$ for $z \in A_\alpha$ and $\alpha(z,i,j) = (\alpha(z,i),j)$ otherwise. Let α_x be such an automorphism for $m = 0$ and $A = \{x\}$. The following claim is easy.

CLAIM: *The group $Aut(M)$ is topologically generated by δ and α_x.*

Notice that the kernel of the cover $M \to W$ consists of continuum automorphisms: for any $A \subseteq \mathbf{Z}$ define an automorphism β_A by $(z,i,j) \to (z,i,j+2)$ for $z \in A$ and $(z,i,j) \to (z,i,j)$ for $z \notin A$. On the other hand, replacing $\mathbf{Z}$ by

$\mathbf{Z}(n)$ in the definition of W and M we obtain finite structures realizing sentences which are true in M. It remains to prove the following claim.

CLAIM: *If a closed subgroup G of the group $Aut(M)$ covers $Aut(W)$, then $G = Aut(M)$.*

Conjugating G by an element of the kernel we can arrange that $\delta \in G$. Let $(x, i) \in W$ and s_x be the corresponding switching. Let σ be an extension of s_x to an automorphism of M belonging to G. By the definition of S_1 we may assume that σ: $(x, i, j) \to (x, 1 - i, j + 1)$ or σ: $(x, i, j) \to (x, 1 - i, j + 3)$ and, if $z \neq x$, then σ: $(z, i, j) \to (z, i, j)$ or σ: $(z, i, j) \to (z, i, j + 2)$. Replacing σ by σ^3 if necessary, we may assume that σ: $(x, i, j) \to (x, 1 - i, j + 1)$.

On the other hand, σ^2 is trivial for all fibres $(y, i) \neq (x, *)$, and sends (x, i, j) to $(x, i, j + 2)$. Conjugating σ^2 by δ we have a similar automorphism in G for any $x \in \mathbf{Z}$. This creates (topologically in G) all automorphisms of the kernel. Multiplying σ by an appropriate automorphism of the kernel we obtain α_x. This finishes the proof. ∎

4.3. FINITE AXIOMATIZABILITY. Notice that the question stated in the beginning of the section is connected with the (absolute) finite axiomatizability issue.

Indeed, assume that a finite cover π: $M \to W$ has a sentence Φ such that (i) $\{\Phi\} \cup Th(W)$ axiomatizes the cover, (ii) if T' is a sufficiently large finite subset of $Th(W)$ and $W' \models T'$, then W' extends to a model of Φ. Then it is clear that if M is finitely axiomatizable, then W is finitely axiomatizable too.

As we noticed above properties (i) and (ii) can be lost in very natural situations. On the other hand, it is unknown if finite axiomatizability of the cover implies finite axiomatizability of the base (at least in the case of $\aleph_1$-categorical theories). We now show that a very similar question for Cayley graphs has the negative answer.

If the Cayley graph Γ_G of a finitely generated group $G = \langle g_1, \ldots, g_n \rangle$ is considered as a unary algebra (with functions $g_i^+(x) = xg_i$ and $g_i^-(x) = xg_i^{-1}$, $i \leq n$), then Γ_G is axiomatizable by sentences of the form $\forall x(t(x) = t'(x))$ or $\forall x(h(x) \neq h'(x))$, where t, t', h, h' are terms. If Γ_G has an axiomatization where the set of sentences of the second form is finite, then we say that Γ_G is **finitely axiomatizable modulo positive sentences**. It is shown in [16], [17] that this condition is equivalent to the existence of a **threading tuple**: a finite set of elements $w_1, \ldots, w_k \in G \setminus \{1\}$ such that every non-trivial cyclic subgroup of G meets the conjugacy class of some w_i.

Example: There exists a finitely generated group G_1 and a homomorphism $G_1 \to G$ inducing a finite cover of the corresponding Cayley graphs, where Γ_{G_1} is finitely axiomatizable modulo positive sentences, but Γ_G is not. Let G_1 be the group defined in Theorem 31.5 of the book [22]. The group G_1 has a central cyclic subgroup K of order n and G_1/K is canonically isomorphic to $G = B(m,n)$, the free m-generated group satisfying $x^n = 1$ (for appropriate n and m the group G is infinite). If $g \in G_1 \setminus K$ then the power g^n belongs to $K \setminus \{1\}$. This means that the elements of K form a threading tuple and Γ_{G_1} is finitely axiomatizable modulo positive sentences. Since K is central in G_1, the action of K on Γ_{G_1} preserves each fibre gK (identified with the corresponding subset of the Cayley graph). It is clear that the natural map $\Gamma_{G_1} \to \Gamma_G$ is a cover with kernel K. Theorem 39.2 of [22] implies that for any finite sequence $g_1, \ldots, g_l \in G$ there exists a non-trivial normal subgroup not containing $g_i, i \leq l$. This obviously implies that G does not have a theading tuple and the structure Γ_G is not finitely axiomatizable modulo positive sentences. ∎

Notice that such an example is impossible in the direction *base* $\to$ *cover*. Indeed, for a surjective homomorphism ρ: $G_1 \to G$ with finite $H = Ker\rho$, a threading tuple for G_1 is just the union of the cosets gH forming a threading tuple of G.

This can be developed as follows. Let π: $\Gamma_1 \to \Gamma_G$ be a finite cover of Γ_G, where Γ_1 is a transitive structure of bounded valency and $Ker\pi$ is finite. Then π induces a surjective homomorphism ρ: $G_1 = Aut(\Gamma_1) \to G$ with finite $H = Ker\rho$. If Γ_G is finitely axiomatizable modulo positive sentences then, as we have already noticed, G_1 has a threading tuple.

We now show that if the base Γ_G is finitely axiomatizable, then so is the cover.

It follows from [1] (or [16], Theorem 3) that the Cayley graph of a finitely generated group G is finitely axiomatizable if and only if G is finitely presented and has a threading tuple. Assume that G is such a group. Since H is finite, G_1 has a threading tuple. A finite set of relations that determine G_1 is obtained from the words determining G (including $x \cdot x^{-1}$) by interposing letters from H for all appropriate substitutions. We also add all relations of H. As a result we have that G_1 is finitely presented. Since $G_1 = Aut(\Gamma_1)$ and G_1 is countable, the stabiliser of a point must be finite. This means that there is a number t such that the G_1-stabilizer of a t-ball is trivial. For this t we define $\hat{\Gamma}_1$ as in Section 1.3. Then the action of G_1 on $\hat{\Gamma}_1$ is regular. This yields that the Cayley graph of G_1 is interdefinable with $\hat{\Gamma}_1$. The former is finitely axiomatizable by [1]. ∎

5. Nice enumerations of non-disintegrated structures

5.1. NICE ENUMERATION OF STRONGLY MINIMAL GROUPS. It is shown in [18] that any disintegrated strongly minimal structure of a finite language can be considered as a structure of bounded valency. Thus many results of the paper work for disintegrated strongly minimal structures in general. On the other hand, the methods of Section 2 must be substantially modified if one wants to use them in the non-disintegrated case. The reason for that is the following fact (where we admit that a group can have some extra structure).

THEOREM 5.1: *Let G be a locally modular, strongly minimal group having a nice enumeration; then G is ω-categorical.*

Proof: Let G_0 be the subgroup of all algebraic elements. Since G has a nice enumeration, G_0 is finite. It suffices to show that the algebraic closure in G is locally finite; then any countable model of $Th(G)$ has a countable dimension and by a theorem of Baldwin and Lachlan $Th(G)$ is countably categorical. It is known that G is abelian and the algebraic closure of G is the linear dependence with respect to the division ring $D(G,\emptyset)$ of $acl^{eq}(\emptyset)$-definable quasi-endomorphisms ([23], Chapter 4). Such quasi-endomorphisms define endomorphisms of G/G_0 and G/G_0 is a (left) vector space over $D(G,\emptyset)$. So we should prove that $D(G,\emptyset)$ is finite.

Suppose that $D(G,\emptyset)$ is infinite. We start with the case when there is a tuple $\bar{\mathbf{b}} \in acl^{eq}(\emptyset)$ such that the elements of $D(G,\emptyset)$ definable over $\bar{\mathbf{b}}$ form an infinite subset (denoted by $D(\bar{\mathbf{b}})$). It is easy to see that $D(\bar{\mathbf{b}})$ is a subring which is a division ring too. We now need the following claim.

CLAIM: *The multiplicative group of an infinite division ring K does not satisfy the ascending chain condition for subgroups.*

Proof: Indeed, in the case of characteristic 0 the field of rationals is embedded into the center of K. It is well-known (and easy) that $\mathbf{Q}^*$ has an infinite ascending chain of the type: $\mathbf{Z} < \mathbf{Z}^2 < \mathbf{Z}^3 < \cdots$.

Let K be of characteristic p and $\mathbf{F}_p$ be the prime subfield. If K is not algebraic over $\mathbf{F}_p$, then $\mathbf{F}_p(x)$ is embedded into K. The group $\mathbf{F}_p(x)^*$ has an infinite ascending chain of subgroups too (using primes of $\mathbf{F}_p[x]$).

Let K be algebraic over $\mathbf{F}_p$. Since $\mathbf{F}_p$ is in the centre, for any $c \notin \mathbf{F}_p$ the subring $\mathbf{F}_p[c]$ is finite (and is a commutative division ring by Proposition 7.1.2 of [19]). By a theorem of Kaplansky (that the multiplicative group of a non-commutative division ring is not periodic, Corollary to Theorem 7.12.3 in [19])

K is commutative. Since an infinite abelian locally finite group has an ascending chain of subgroups, the claim is proved. ∎

Since $D(\bar{\mathbf{b}})$ is infinite, by the claim we find $H_0 < H_1 < \cdots$, an infinite chain of subgroups of $D(\bar{\mathbf{b}})^*$. Let $\bar{\mathbf{b}}_l, l \leq s$, be the $Aut(G)$-conjugates of $\bar{\mathbf{b}}$ and H_i^l consist of the corresponding conjugates of quasi-endomorphisms from H_i.

Let F be a field. Consider the right permutation module FG over $FAut(G)$. By [3] it suffices to show that FG does not satisfy the ascending chain condition for submodules. For each $i \in \omega$ define

$$V_i = \{\Sigma_{j\in\delta} a_j w_j \in FG\colon \text{there is a decomposition}$$
$$\Sigma_{j\in\delta} a_j w_j = \Sigma_{j\in\delta} a_{1j} w_j + \cdots + \Sigma_{j\in\delta} a_{sj} w_j \text{ such that}$$
$$(\forall l \leq s)(\forall k \in \delta)(\Sigma\{a_{lj}\colon (w_j + G_0) \in H_i^l(w_k + G_0)\} = 0)\}.$$

We admit here that some a_{lj} can be 0. Notice that V_i is a (right) submodule of FG. Indeed, the closedness under $+$ is obvious. On the other hand, if $a\phi \in FAut(G)$ and $\Sigma_\delta a_j w_j \in V_i$ then $(\Sigma_\delta a_j w_j)(a\phi) = \Sigma_\delta a a_j (w_j)\phi$ and for any $i, k \in \delta$, $(w_j)\phi \in H_i^m(w_k)\phi \leftrightarrow w_j \in H_i^l w_k$, where $\bar{\mathbf{b}}_l\phi = \bar{\mathbf{b}}_m$. This guarantees that $(\Sigma_\delta a_j w_j)(a\phi) \in V_i$.

It is easily seen that $V_i \subset V_{i+1}$ and $V_i \neq V_{i+1}$. This yields that the ascending chain condition for right submodules in FG does not hold, a contradiction with [3].

Consider the case when for all tuples $\bar{\mathbf{b}} \in acl^{eq}(\emptyset)$ the subring $D(\bar{\mathbf{b}})$ is finite (then $D(\bar{\mathbf{b}})$ is a field). Find an infinite ascending chain of subfields $D(\bar{\mathbf{b}}_i)$ (we may assume that every $\bar{\mathbf{b}}_i$ is a subtuple of $\bar{\mathbf{b}}_{i+1}$) and their conjugates $D(\bar{\mathbf{b}}_i^l)$ (now the number s_i of conjugates depends on i). We define submodules V_i as above:

$$V_i = \{\Sigma_{j\in\delta} a_j w_j \in FG\colon \text{there is a decomposition}$$
$$\Sigma_{j\in\delta} a_j w_j = \Sigma_{j\in\delta} a_{1j} w_j + \cdots + \Sigma_{j\in\delta} a_{s_i j} w_j \text{ such that}$$
$$(\forall l \leq s)(\forall k \in \delta)(\Sigma\{a_{lj}\colon (w_j + G_0) \in D(\bar{\mathbf{b}}_i^l)^*(w_k + G_0)\} = 0)\}.$$

The condition $V_i \subset V_{i+1}$ follows from the fact that quasi-endomorphisms definable over $\bar{\mathbf{b}}_i^l$ are definable over any extension of the form $\bar{\mathbf{b}}_{i+1}^m$; so a decomposition guaranteeing $\Sigma_{j\in\delta} a_j w_j \in V_i$ also guarantees $\Sigma_{j\in\delta} a_j w_j \in V_{i+1}$. The rest is as above. ∎

This theorem shows that the method of nice enumerations cannot be applied in many natural situations. *Is there any non-disintegrated $\aleph_1$-categorical but not $\aleph_0$-categorical structure having a nice enumeration?* The theorem above suggests that *if M is a locally modular $\aleph_1$-categorical structure, then the answer* *is negative.* A natural way of proving this is to show that some strongly minimal set interpretable in M has locally finite algebraic closure. Since strongly minimal sets are non-orthogonal, then we would see that its dimensions in any countable model of $Th(M)$ are countable and by a theorem of Baldwin and Lachlan $Th(M)$ is countably categorical.

Buechler's coordinatization theorem (Proposition 2.5.9 in [23]) implies that for any $a \in M$ there exists $c \in acl^{eq}(a)$ of rank 1. Let D be a strongly minimal set determined by $tp(c/\emptyset)$. By a theorem of Hrushovski (Proposition 5.2.4 in [23]) the geometry associated to p is projective or affine over a division ring. In the latter case by Proposition 5.2.3 from [23] there exists a minimal group G $acl(\emptyset)$-definable in D^{eq} and the algebraic closure of G is the linear dependence with respect to the division ring of $acl(\emptyset)$-definable quasi-endomorphisms.

However, the argument of the theorem above does not work for projective or affine spaces (for example, Corollary 3.6 of [3] states that the permutation module of 1-dimensional affine space over the rationals satisfies the acsending chain condition for submodules). Moreover, we cannot even guarantee that D and G above have nice enumerations. Indeed, it is not clear how to define a nice enumeration for a structure definable in M^{eq} (it can be difficult even in M^k if we do not have a length function as in Section 3). The following questions arise in these considerations:

- Is there a projective space over an infinite division ring having a nice enumeration?

- Is there an affine space over an infinite division ring having a nice enumeration?

Some partial answers are given in Section 5.2.

We finish this subsection with a fact concerning the case when the structure is not locally modular. This has been found by the referee. We need the following definitions from [15]. Let T be an $\aleph_1$-categorical theory and A, B be subsets of the universal domain $\mathbf{C}$. The set B is **finitely generated over** A if $B = dcl(A \cup B_0)$ for some finite B_0. The set B is **normal over** A if it is invariant with respect to $Aut(\mathbf{C}/A)$. A finite simple group S is **involved in** T if there are $A, B \subseteq \mathbf{C}$ with B finitely generated and normal over A, such that G is a composition factor of $Aut(B/A)$. If additionally A is finitely generated over a model of T then we say that G is strongly involved in T. In [15] Hrushovski has proved that if T *is not locally modular then alternating groups* Alt_n *of arbitrary large rank are strongly involved in* T. Using the method of this theorem we shall prove the following proposition.

PROPOSITION 5.2: *Let M be a countably saturated, non-locally modular, strongly minimal structure. Then M does not have a nice enumeration.*

Proof: Since M is not ω-categorical (otherwise it is locally modular, [23], Chapter 2), the algebraic closure on M is not locally finite. This means that we may assume that there exists a tuple $\bar{c}$ in M such that $acl(\bar{c})$ is infinite. Adding some parameters from $acl(\bar{c})$ and repeating the proof of Proposition A in [15] (and Section 5.3.2 from [23]) we can show that for any n there exists a strongly minimal subset $C(\bar{b}) \subseteq M^2$, defined over $\bar{b}\bar{c}$ with $rk(\bar{b}/\bar{c}) = n$ such that if $\bar{b}'$ is an independent from $\bar{b}$ realization of $tp(\bar{b}/\bar{c})$ then the following properties hold:

(a) the structure $(C(\bar{b}) \cap C(\bar{b}'), acl)$ is a pregeometry of dimension $n-1$ or n,

(b) there exists an independent subset X of $C(\bar{b}) \cap C(\bar{b}')$ of size $n-1$ such that $Aut(M/\bar{c}\bar{b}\bar{b}', \{X\})$ induces $Sym(X)$ on X.

Take $\bar{b}'$ as above. Extending $\bar{c}$ by $\bar{b}$ and some subtuple $\bar{b}'' \subset \bar{b}'$ with $rk(\bar{b}''/\bar{c}) = n-1$ we obtain a tuple $\bar{u}_n$, and a formula $\phi_n(\bar{u}_n, x, y)$ such that for any t, non-algebraic over $\bar{u}_n$, the set of all realizations of $\phi_n(\bar{u}_n, t, y)$ is a finite set which is transitive with respect to $Aut(M/\bar{u}_n t)$ and contains a subset X of size $n-1$ such that $Aut(M/\bar{u}_n t, \{X\})$ induces $Sym(X)$ on X (we may also assume that any element of X is not algebraic over $\bar{u}_n$). Then $Aut(M/\bar{u}_n t)$ has a composition factor isomorphic to Alt_{n-1}.

Suppose that M has a nice enumeration. For $n \in \omega$ take the nice subset $w_1, \ldots, w_k$ containing $\bar{u}_n$ as above such that w_k is the first element non-algebraic over $\bar{u}_n$. Let V_k be the set of realizations of $\phi_n(\bar{u}_n, w_k, y)$. Then the least element of V_k is enumerated after w_k. By strong minimality if the group induced by $Aut(M/w_1, \ldots, w_k, w_{k+1})$ on V_k does not have Alt_{n-1} as a composition factor, then $w_{k+1} \in acl(w_1, \ldots, w_k)$. Let V'_k be the orbit of w_{k+1} with respect to $Aut(M/w_1, \ldots, w_k)$. Let G be the group of all permutations on $V_k \times V'_k$ induced by $Aut(M/w_1, \ldots, w_k)$. Since the pointwise stabilizer of V'_k is a normal subgroup of G without Alt_{n-1} as a composition factor (and G/G_{V_k} has Alt_{n-1} as a composition factor), the group induced by G on V'_k ($\cong G/G_{V'_k}$) induces Alt_{n-1} as a composition factor.

If the group induced by $Aut(M/w_1, \ldots, w_k, w_{k+1})$ on V_k has Alt_{n-1} as a composition factor, then we consider if this holds for the group $Aut(M/w_1, \ldots, w_k, w_{k+1}, w_{k+2})$. Continuating this argument we find a nice subset $w_1, \ldots, w_k, \ldots, w_l$ such that $w_{l+1} \in acl(w_1, \ldots, w_l)$ and the group induced by $Aut(M/w_1, \ldots, w_l)$ on the corresponding orbit of w_{l+1} has Alt_{n-1} as a composition factor. Now find infinite sequences $l_1 < l_2 < \cdots$ and $n_1 < n_2 < \cdots$

such that any $Aut(M/w_1, \dots, w_{l_i})$ induces on the corresponding orbit of w_{l_i+1} a group having Alt_{n_i} as a composition factor and without composition factors isomorphic to Alt_{n_j}, $i < j$. It is easy to see that the pairs $(\{w_1, \dots, w_{l_i}\}, w_{l_i+1})$ form an anti-chain contradicting nice enumerability. ∎

5.2. NICE ENUMERATIONS OF AFFINE AND PROJECTIVE SPACES. Let D be a division ring, $n \in \omega$ and $AGL(n, D)$ be the group of affine transformations of the right vector space D^n, i.e., the permutations of the form $\bar{x} \to \bar{x}A + \bar{a}$, where A is a matrix over D and $\bar{x}A$ is defined by matrix multiplication. Let $\Gamma L(n, D)$ be the group of all semi-linear transformations of D^n and $H(n, D)$ be the group of all scalar matrices in $\Gamma L(n, D)$ (considered as semi-linear transformations with respect to the corresponding inner automorphism of D). The group $GL(n, D)$ of all linear transformations of D^n is defined as the centralizer of $H(n, D)$ in $\Gamma L(n, D)$. Linear transformations are considered as matrices. Then the action of the group $PGL(n, D) = GL(n, D)/Z(GL(n, D))$ on the right projective space $P_{n-1}(D)$ is also defined by right multiplication of matrices (since $GL(n, D)$ centralizes $H(n, D)$, the definition is correct).

We start with the case when D is the union of an infinite chain of division rings. The following general lemma covers this case for both spaces. It has been suggested by the referee. The proof generalizes the previous argument of the author in the case of affine spaces.

LEMMA 5.3: *Let a permutation structure (G, X) be the union of a sequence of substructures (G_i, X_i) with $X_i \subseteq X_{i+1}$ and $G_i = G_{\{X_i\}}$. Assume that there are $a_1, \dots, a_m \in X_1$ such that for every i the condition $\bigwedge_{1 \le j \le m} g(a_j) \in X_i$ implies $g \in G_i$. Then (G, X) does not have a nice enumeration.*

Proof: Let $x_1, x_2, \dots, x_k, \dots$ be an enumeration of X. Let B be the set of all elements b such that some X_i includes all predecessors of b in the enumeration but $b \in X_{i+1} \setminus X_i$. We may assume that all $b \in B$ are enumerated after $a_1, \dots, a_m$. It is easy to see that B is infinite.

Suppose that $b_1 \in B$ is enumerated before $b_2 \in B$ and X_t has the property that $b_2 \in X_{t+1} \setminus X_t$. If $g \in G$ takes b_1 to b_2 then g takes some a_l with $l \le m$ to some X_s with $t < s$. Since $g(a_l)$ has the number greater than the number of b_2 we see that the nice pair defined by b_1 cannot be mapped to the nice pair defined by b_2. This shows that there is an infinite anti-chain of nice pairs. ∎

THEOREM 5.4: *Let D be the union of an infinite chain of division rings $F_0 < \dots < F_i < \cdots$. Then the permutation structures $(AGL(n, D), D^n)$ and*

$(PGL(n,D), P_{n-1}(D))$ do not have a nice enumeration.

Proof: To apply the lemma in the case of affine spaces consider $(AGL(n,D), D^n)$ as the union of spaces $(AGL(n,F_i), F_i^n)$. Let $r \in F_1 \setminus \{0,1\}$ and $a_1 := (1,0,\ldots,0)$, $a_2 := (0,1,\ldots,0)$,..., $a_n := (0,0,\ldots,1)$, $a_{n+1} := (r,0,\ldots,0)$, $a_{n+2} = (0,r,\ldots,0)$,..., $a_{2n} = (0,0,\ldots,r)$. Now it is easy to see that if for a transformation $\bar{x} \to \bar{x}A + \bar{c}$ the matrix A or the vector $\bar{c}$ contains an element from $D \setminus F_i$, then one of the vectors $(0,\ldots,1,\ldots,0)A+\bar{c}$ or $(0,\ldots,r,\ldots,0)A+\bar{c}$ (for an appropriate place) contains an element of $D \setminus F_i$.

In the case of projective spaces notice that any space $P_{n-1}(F_i)$ can be naturally embedded into $P_{n-1}(D)$ by a map which assigns to a 1-dimensional subspace of F_i^n its extension to a 1-dimensional subspace of D^n. To describe groups $G_i = G_{\{X_i\}}$ (where all $P_{n-1}(F_i)$ are considered as X_i) notice that $Z(GL(n,D)) = Z(H(n,D))$. Then the subgroup

$$PGL(n,F_i)^D := GL(n,F_i)Z(H(n,D))/Z(GL(n,D)) < PGL(n,D)$$

has a natural (right) action on $P_{n-1}(F_i)$. Consider $(PGL(n,D), P_{n-1}(D))$ as the union of spaces $(PGL(n,F_i)^D, P_{n-1}(F_i))$. Let $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ be subspaces of D^n generated by $(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1)$ respectively. Let the next $\binom{n}{2}$ elements $\bar{a}_{n+1}, \bar{a}_{n+2}, \ldots$ denote all subspaces generated by elements of the form $(0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0)$.

To verify the assumptions of the lemma consider a transformation $\bar{x} \to \bar{x}B$ such that for any scalar matrix C over $Z(D)$, $BC \notin GL(n,F_i)$. Then there are $b_{j,k}, b_{l,m} \in B$ such that for any $d \in Z(D)$, $\{b_{j,k}d, b_{l,m}d\} \not\subset F_i$. Indeed, find $b_{j,k}, b_{l,m} \in B$ such that one of the following conditions holds: for any $d \in Z(D)$, $b_{j,k}d, b_{l,m}d \notin F_i$ or there is $d \in Z(D)$ with $b_{j,k}d \in F_i$ and $b_{l,m}d \notin F_i$ (if in the latter case there is $d' \in Z(D)$ with $\{b_{j,k}d', b_{l,m}d'\} \subset F_i$ then $(d')^{-1}d = (b_{j,k}d')^{-1}b_{j,k}d \in F_i$ and $b_{l,m}d = b_{l,m}d'(d')^{-1}d \in F_i$). If $j = l$, then $\bar{a}_jB \notin P_{n-1}(F_i)$ with respect to the embedding into $P_{n-1}(D)$. If $j \neq l$ and $\bar{a}_mB \in P_{n-1}(F_i)$ for all $1 \leq m \leq n$, then there is an element $\bar{a}_{n+t}B$, $1 \leq t$, which does not belong to $P_{n-1}(F_i)$. ∎

COROLLARY 5.5: *If D is not finitely generated over the prime subfield (for example, has an infinite transcendental degree over the prime subfield), then the permutation structures $(AGL(n,D), D^n)$ and $(PGL(n,D), P_{n-1}(D))$ do not have a nice enumeration.*

The corollary shows that the case when D is a division ring finitely generated over the prime subfield is basic in the problem. In the affine case we can find some further restrictions in terms of valuations.

A **valuation of a field** K **into** $\mathbf{R} \cup \{\infty\}$ is a map $v: K \to \mathbf{R}$, satisfying

- $v(1) = 0$ and $\{x: v(x) = \infty\} = \{0\}$;
- $v(ab) = v(a) + v(b)$;
- $v(a + b) \geq min(v(a), v(b))$.

A family of valuations Φ of K has the **strong approximation property**, if for any finite subfamily $\{v_1, \dots, v_n\} \subseteq \Phi$ and any sequences $a_1, \dots, a_n \in K$, and $\epsilon_i \in v_i(K)$, $1 \leq i \leq n$, there exists $a \in K$ such that $v_i(a - a_i) = \epsilon_i$ and $v(a) \geq 0$ for any $v \in \Phi \setminus \{v_1, \dots, v_n\}$. In the commutative case, if R is a Dedekind domain, then there exists a family Φ of discrete valuations of the field of quotients K having the strong approximation property such that R is the intersection of the corresponding family of valuation rings and Φ is *almost finite*: for any $a \in K$ the set of $\phi \in \Phi$ with $\phi(a) \neq 0$ is finite [25]. This motivates the following theorem.

THEOREM 5.6: *Let D be a finite extension of a field K, which is central in D. If K has an infinite almost finite family of non-trivial valuations, then the permutation structure $(AGL(n, D), D^n)$ does not have a nice enumeration.*

Proof: Let $d_1, \dots, d_n$ be a basis of D over K. Since Φ is almost finite there exists a cofinite subfamily $\Phi' \subseteq \Phi$ such that for any pair d_i, d_j and any $\phi \in \Phi'$ if $d_i d_j = c_1 d_1 + \dots + c_n d_n$, $c_i \in K$, then $\phi(c_k) = 0$, $k \leq n$. Take any $r \in K \setminus \{0, 1\}$. We may assume that $\phi(r) = 0$ for all $\phi \in \Phi'$.

Let $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_k, \dots$ be an enumeration of D^n. Let $\bar{b}_1$ be a vector from D^n containing a coordinate s_1 of the form $\sum_{i \leq n} c_i d_i$, where $\phi_1(c_k) < 0$ for some c_k and some $\phi_1 \in \Phi'$. The vector $\bar{b}_1$ can be chosen so that no vector from D^n enumerated in $\bar{a}_1, \dots, \bar{b}_1$ before $\bar{b}_1$ includes an element $s = \sum_{i \leq n} c'_i d_i$ where $\phi_1(c'_i) < 0$ holds for some i. We also assume that the vectors

$$(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1), (r, 0, \dots, 0), (0, r, \dots, 0), \dots, (0, 0, \dots, r)$$

are enumerated before $\bar{b}_1$. Similarly, choose $\bar{b}_2 = (\dots, s_2, \dots)$, $s_2 = \sum_{i \leq n} c''_i d_i$, assuming that for some $\phi_2 \in \Phi'$ and $i \leq n$, $\phi_2(c''_i) < 0$ and $\bar{b}_2$ is the first vector in the enumeration $\bar{a}_1, \dots, \bar{b}_2, \dots$, where ϕ_2 has this property for some coordinate. The existence of such $\bar{b}_2$ and ϕ_2 follows from non-triviality and almost finiteness.

A natural procedure continuing this choice provides an infinite sequence $\bar{b}_1, \dots, \bar{b}_l, \dots$ inducing an infinite anti-chain of nice pairs. Indeed, if a transformation $\bar{x} \to \bar{x}A + \bar{a}$ takes $\bar{b}_i$ onto $\bar{b}_j$ (we may assume that $i < j$), then by the choice of Φ', an element m with $\phi_j(m) < 0$ must occur in the $\bar{d}$-decompositions of the elements of the matrix A or of the vector $\bar{a}$. Then one of the vectors $(0,\ldots,1,\ldots,0)A+\bar{a}$ or $(0,\ldots,r,\ldots,0)A+\bar{a}$ (for an appropriate place) contains a coordinate with $\phi_j(q) < 0$ for some coefficient in the $\bar{d}$-decomposition. This contradicts the choice of $\bar{b}_j$. ∎

COROLLARY 5.7: *Let D be a field. Then the permutation structure $(AGL(n,D), D^n)$ does not have a nice enumeration.*

Proof: If the structure has a nice enumeration, then by Corollary 5.5 there are two possibilities: $char(D) = 0$ and D is a finite extension of some $\mathbf{Q}(X_1,\ldots,X_n)$, $0 \le n$, or D is a finite extension of some $\mathbf{F}_p(X_1,\ldots,X_n)$, $1 \le n$. The cases are considered in the same way. The ring $K(X_1,\ldots,X_{n-1})[X_n]$ is a Dedekind domain and now Theorem 5.6 works. If D is a finite extension of $\mathbf{Q}$ then Theorem 5.6 is applied to the family of p-valuations, $p \in \pi_\infty$. ∎

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